Comparison of Quality Power Spectrum Estimation (Bartlett, Welch, Blackman & Tukey) Methods

Rahul U. Kale, Pavan M. Ingale, Rameshwar T. Murade, Sarfaraz S. Sayyad

***Abstract*— To convert the analog signals into samples of digital signals is called the DSP. In this paper we study the power spectrum estimation of DSP. There is a fluctuation on DSP signals thermal noise in resistors & electronic devices.**

There are two methods for calculating PSE is parametric & non parametric methods. In Nonparametric (classical) methods – begin by estimating the autocorrelation sequence from a given data. Limitations of non parametric methods are we require inherent assumptions for autocorrelation estimate.

Parametric method we assume that signal is output of a system having white noise as an input. We model the system and get its parameters i.e. coloring filter coefficients and predict the power spectrum. Here we compare the Yule-Walker method & Burg method for Power spectrum estimation. In this paper we see that parametric methods do not need these assumptions.

***Keywords*— Periodogram, Bartlett method, Welch method Blackman & Tukey method**

I. INTRODUCTION

There are mainly two types of power spectrum estimation (PSE) methods: parametric and nonparametric. In contrast to parametric methods, non-parametric methods do not make any assumptions on the data-generating process or model (e.g. autoregressive model). This paper analyzes five common non-parametric PSE methods. They are: Periodogram Method, Modified Periodogram Method, Bartlett's Method, Welch's Method, and Blackman-Tukey Method. All these are estimation methods. The ideal power spectrum of a signal x (n) can be computed by first finding the ideal autocorrelation r (x) x :

$$r_x(\alpha) = E[x(n)x(n+\alpha)] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)x(n+\alpha).$$

This is an average computed over the infinite interval. However, in image processing, the signal is often of finite length because the size of any image is finite. Even with a large image, in order to make the assumption of stationary, often only a small window of the image is considered at a time. In this paper, it is assumed that the window is small enough so sthat the assumption of stationary holds, allowing the autocorrelation and the power spectrum to be a Fourier Transform pair. Firstly, the ideal power spectrum is "blurred" or "smoothened" by the low pass filter, and the amount of blurring is mainly determined by the width of the main lobe. Secondly, the power spectrum at any particular frequency is "leaking" into the side lobes, possibly "masking" the power spectra at the nearby frequencies.

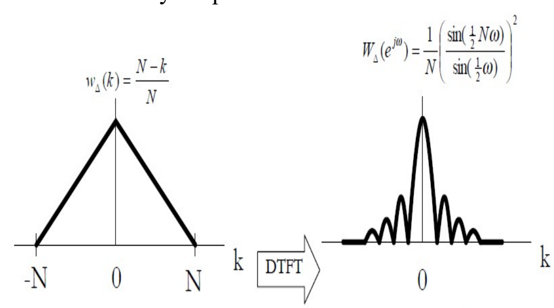

Figure 1 – Triangular Window $w_\Delta(k)$ and its DTFT $W_\Delta(e^{j\omega})$

Bartlett Method: In Bartlett method we divide the signal into blocks, find their period grams and average them to get the Power spectrum.

2. Welch Method: It is same method of Bartlett with some modifications data segments can be overlapping. This method has got better precision but less frequency resolution than Bartlett method.

3. Blackman-Tukey Method: In this method we windowed the auto-correlation sequence and take Fourier transform to get power spectrum estimate (Periodogram) in effect we smooth out the Periodogram.

Also compare these three methods according to characteristics asymptotic bias, variance, complexity & spectral resolution.

All these methods no assumption about how the data were generated & hence called nonparametric methods. In this paper we have five sections in first we introduce the periodogram and next three sections we have explain all three methods with equations. Five sections we tabulate the comparison of methods with quality factor.

II. BARTLETT METHOD

In this method reducing the variance in the periodogram involves three steps. First the N-point sequence sub divided into K non overlapping segments. [2]

Where each segment has length M.

$$x_i(n) = x(n+iM) -----(1)$$

$$i = 0,1......., K-1$$

$$n = 0,1......, M-1$$

For each segment, we compute the periodogram,

$$P_{xx}^{(i)}(f) = \frac{1}{M}\left|\sum_{n=0}^{M-1} x_i(n)\,\mathrm{e}^{-j2\Pi fn}\right|^2 ----(2)$$

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Finally, we average the periodogram for the K segment to obtain,

$$P_{xx}^{B}(f)=\frac{1}{K}\sum_{i=0}^{K-1}P_{xx}^{(i)}(f)----(3)$$

The statistical properties of this estimate are easily obtained. First the mean value is,

$$E[\mathrm{P}_{xx}^{B}(\mathrm{f})]=\frac{1}{K}\sum_{i=0}^{K-1}E[P_{xx}^{(i)}(f)]----(4)$$

$$=E[P_{xx}^{(i)}(f)]$$

The value of single periodogram as,

$$E[P_{xx}^{(i)}(f)]=\sum_{m=-(M-1)}^{M-1}(1-\frac{|m|}{M})\gamma_{xx}(m)e^{-j2\pi fm}----(5)$$

$$=\frac{1}{M}\int_{-1/2}^{1/2}\tau_{xx}(\alpha)\left(\frac{\sin\pi(f-\alpha)M}{\sin\pi(f-\alpha)}\right)^{2}d\alpha$$

Where

$$W_B(f)=\frac{1}{M}\left(\frac{\sin\pi fM}{\sin\pi f}\right)^{2}----(6)$$

The frequency characteristics of Bartlett window,

$$W_B=\begin{cases}1-\frac{|m|}{M}, \\ 0,\end{cases}$$

$$|m|\le M-1------(7)$$

From equation (5) we observe that the true spectrum is now convolved with the frequency characteristics WB (f) of Bartlett window.

We have to reduced the variance .The variance of Bartlett estimate is,

$$\mathrm{var}[\mathrm{P}_{xx}^{B}(f)]=\frac{1}{K^{2}}\sum_{i=0}^{K-1}\mathrm{var}[P_{xx}^{(i)}]$$

$$\mathrm{var}[\mathrm{P}_{xx}^{B}(f)]=\frac{1}{K}\mathrm{var}[P_{xx}^{(i)}]----------(8)$$

We obtain the,

$$\mathrm{var}[P_{xx}^{B}(\mathrm{f})]=\frac{1}{K}\tau_{xx}^{2}(f)\left[1+\left(\frac{\sin 2\pi fM}{M\sin 2\pi f}\right)^{2}\right]----(9)$$

Therefore, the variance of Bartlett power spectrum estimate has been reduced by the factor K.

A. *Performance characteristics of this Method for PSE*

We observe that the Bartlett PSE is asymptotically unbiased & if K is allowed to increase with an increase in N, the estimate is also consistent. [2]

Hence, asymptotically, this estimate is characterized by the quality factor,

$$Q_B=K=\frac{N}{M}----(43)$$

$$\Delta f=\frac{0.9}{M}$$

$$Q_B=\frac{N}{0.9/\Delta f}=1.1N\Delta f----(44)$$

III. WELCH METHOD

In this method two basic modification to Bartlett method. First, he allowed the data segments to overlap. Second is to window the data segments prior to computing the periodogram. [1]

The data segments to can be represented as,

$$x_i(n)=x(n+iD)-----(10)$$

$$n=0,1.......,\mathrm{M}-1$$

$$i=0,1......,\mathrm{L}-1$$

Where iD is starting point for the i_{th} sequence.

If D=M Segment do not overlap, L data segments identical to the Number K in Bartlett method.

If D=M/2, 50% overlap.

The modified periodogram is,

$$\overline{P}_{xx}^{(i)}(f)=\frac{1}{MU}\left|\sum_{n=0}^{M-1}x_i(n)\omega(n)e^{-2j\pi fn}\right|^{2}-----(11)$$

$$where, i=0,1,......., L-1$$

Where U is normalization factor for the power in the window function ,

$$U=\frac{1}{M}\sum_{n=0}^{M-1}\omega^{2}(n)----(12)$$

Welch PSE is average of these modified periodogram,

$$P_{xx}^{(\mathrm{w})}(f)=\frac{1}{L}\sum_{i=0}^{L-1}\overline{P}_{xx}^{(i)}(f)----(13)$$

The mean value of Welch estimate is

$$E\left|P_{xx}^{w}(f)\right|=\frac{1}{L}\sum_{i=0}^{L-1}E\left[\overline{P}_{xx}^{(i)}(f)\right]$$

$$E\left|P_{xx}^{w}(f)\right|=E\left[\overline{P}_{xx}^{(i)}(f)\right]----(14)$$

Expected value of modified periodogram is

$$E\left[\overline{P}_{xx}^{(i)}(f)\right]=\frac{1}{MU}\sum_{n=0}^{M-1}\sum_{m=0}^{M-1}\omega(n)\omega(m)E\left[x_i(n)x_i^{*}(m)\right]e^{-j2\pi f(n-m)}----(15)$$

$$E\left[\overline{P}_{xx}^{(i)}(f)\right]=\frac{1}{MU}\sum_{n=0}^{M-1}\sum_{m=0}^{M-1}\omega(n)\omega(m)\gamma_{xx}(n-m)e^{-j2\pi f(n-m)}$$

$$\sin ce$$

$$\gamma_{xx}(n)=\int_{-1/2}^{1/2}\tau_{xx}(\alpha)e^{j2\pi\alpha n}----(16)$$

From equation (15) & (16) we get

$$E\left[\overline{P}_{xx}^{(i)}(f)\right]=\frac{1}{MU}\int_{1/2}^{-1/2}\tau_{xx}(\alpha)\left[\sum_{n=0}^{M-1}\sum_{m=0}^{M-1}\omega(n)\omega(m)\gamma_{xx}(n-m)e^{-j2\pi f(n-m)(f-\alpha)}\right]d\alpha---(17)$$

$$E\left[\overline{P}_{xx}^{(i)}(f)\right]=\int_{1/2}^{-1/2}\tau_{xx}(\alpha)w(f-\alpha)d\alpha----(18)$$

Where by definition

$$w(f)=\frac{1}{MU}\left|\sum_{n=0}^{M-1}\omega(n)e^{-j2\pi fn}\right|^{2}----(19)$$

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The normalization factor U ensures that

$$\int_{-1/2}^{1/2} w(f)df = 1----(20)$$

The variance of Welch method is

$$\mathrm{var}\left[P_{xx}^{w}(f)\right]=\frac{1}{L^2}\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}E\left[\overline{P}_{xx}^{(i)}(f)\overline{P}_{xx}^{(j)}(f)\right]-\left\{E[P_{xx}^{w}(f)]\right\}^2----(21)$$

In the case of no overlap between successive data segments,

$$\mathrm{var}\left[P_{xx}^{w}(f)\right]=\frac{1}{L}\mathrm{var}\left[\overline{P}_{xx}^{(i)}(f)\right]$$

$$\mathrm{var}\left[P_{xx}^{w}(f)\right]\approx\frac{1}{L}\tau_{xx}^{2}(f)----(22)$$

In the case 50% overlap between successive data segment,

$$\mathrm{var}\left[P_{xx}^{w}(f)\right]\approx\frac{9}{8L}\tau_{xx}^{2}(f)----(23)$$

A. Performance characteristics of this Method for PSE

Under the two conditions given by the quality factor is,
L=N/M for no overlap [2]
8L/9=17N/9M for 50 % overlap.
The spectral width of triangular window at the 3-db points is

$$\Delta f=\frac{1.28}{M}----(45)$$

Quality factor expressed in terms of N &Δ f,

$$Q_w=\begin{cases}0.78N\Delta f\\ 1.39N\Delta f\end{cases}$$

IV. BLACKMAN & TUKEY METHOD

In this method sample the autocorrelation sequence is windowed first & then Fourier transformed to yield the estimate of power spectrum.

$$P_{xx}^{BT}(f)=\sum_{m=-(M-1)}^{M-1}\tau_{xx}(m)w(m)e^{-j2\pi fm}----(24)$$

Where the window function w(n) has length 2M-1 & is zero Hence the frequency domain equivalent expression for this,

$$P_{xx}^{BT}(f)=\int_{-1/2}^{1/2}P_{xx}(\alpha)w(f-\alpha)d\alpha----(25)$$

The window sequence w (n) should symmetric (even) about m=0 to ensure that the estimate of power spectrum is real.

$$w(f)\geq 0\,|f|\leq 1/2---(26)$$

Blackman & Tukey PSE,

$$E\left[P_{xx}^{BT}(f)\right]=\int_{-1/2}^{1/2}E\left[P_{xx}(\alpha)\right]w(f-\alpha)----(27)$$

We have

$$E\left[P_{xx}^{BT}(f)\right]=\int_{-1/2}^{1/2}\tau_{xx}(\theta)W_B(\alpha-\theta)d\theta----(28)$$

From equation 26 & 27 double convolution integral is,

$$E\left[P_{xx}^{BT}(f)\right]=\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\tau_{xx}(\theta)W_B(\alpha-\theta)\mathrm{w}(\mathrm{f}-\alpha)d\alpha d\theta----(29)$$

The working time domain is,

$$E\left[P_{xx}^{BT}(f)\right]=\sum_{m=-(M-1)}^{M-1}E\left[\tau_{xx}(m)\right]\omega(m)e^{-j2\pi fm}$$

$$E\left[P_{xx}^{BT}(f)\right]=\sum_{m=-(M-1)}^{M-1}\gamma_{xx}(m)\omega_B(m)\omega(m)e^{-j2\pi fm}----(30)$$

The Bartlett window is,

$$w_B(n)=\begin{cases}1-\frac{|m|}{M}\\ 0\end{cases}$$

$$|m|\leq M-1$$

The additional smoothing of periodogram
Since

$$\int_{-1/2}^{1/2}w_B(\alpha-\theta)(\mathrm{f}-\alpha)d\alpha=\int_{-1/2}^{1/2}w_B(\alpha)\mathrm{w}(\mathrm{f}-\theta-\alpha)d\alpha----(33)$$

$$\int_{-1/2}^{1/2}w_B(\alpha-\theta)(\mathrm{f}-\alpha)d\alpha\approx w(f-\theta)$$

Variance of Blackman & Tukey PSE,

$$\mathrm{var}\left[P_{xx}^{BT}(f)\right]=E\left\{\left[P_{xx}^{BT}(f)^2\right]\right\}-\left\{E\left[P_{xx}^{BT}(f)\right]\right\}^2----(34)$$

Second moment in equation (34)

$$E\left\{\left[P_{xx}^{BT}(f)^2\right]\right\}=\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}E\left[P_{xx}(\alpha)P_{xx}(\theta)\right]w(f-\alpha)w(f-\theta)d\alpha d\theta----(35)$$

The random process is Gaussian find that,

$$E\left[P_{xx}(\alpha)P_{xx}(\theta)\right]=\tau_{xx}(\alpha)\tau_{xx}(\theta)\left\{1+\left[\frac{\sin\pi(\theta+\alpha)N}{N\sin\pi(\theta+\alpha)}\right]^2+\left[\frac{\sin\pi(\theta-\alpha)N}{N\sin\pi(\theta-\alpha)}\right]^2\right\}----(36)$$

From equation (35) & (36)

$$E\left\{\left[P_{xx}^{BT}(f)^2\right]\right\}=\left[\int_{-1/2}^{1/2}\tau_{xx}(\theta)w(f-\theta)d\theta\right]^2+$$

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\tau_{xx}(\theta)\,\mathrm{w}(\mathrm{f}-\theta)\,\mathrm{x}$$

$$\left\{1+\left[\frac{\sin\pi(\theta+\alpha)N}{N\sin\pi(\theta+\alpha)}\right]^2+\left[\frac{\sin\pi(\theta-\alpha)N}{N\sin\pi(\theta-\alpha)}\right]^2\right\}d\alpha d\theta----(37)$$

From equation (37) & (33),

$$\mathrm{var}\left[P_{xx}^{BT}(f)^2\right]\approx\frac{1}{N}\int_{-1/2}^{1/2}\tau_{xx}(\alpha)w(f-\alpha)\left[\tau_{xx}(-\alpha)w(f+\alpha)+\tau_{xx}(\alpha)w(f-\alpha)\right]d\alpha$$

$$\mathrm{var}\left[P_{xx}^{BT}(f)^2\right]\approx\frac{1}{N}\int_{-1/2}^{1/2}\tau_{xx}^2(\alpha)\mathrm{w}^2(f-\alpha)d\alpha----(39)$$

. Approximation in variance is

$$\mathrm{var}\left[P_{xx}^{BT}(f)^2\right]\approx\frac{1}{N}\int_{-1/2}^{1/2}\tau_{xx}(\alpha)w(f-\alpha)\left[\tau_{xx}(-\alpha)w(f+\alpha)+\tau_{xx}(\alpha)w(f-\alpha)\right]d\alpha$$

$$\mathrm{var}\left[P_{xx}^{BT}(f)^2\right]\approx\frac{1}{N}\int_{-1/2}^{1/2}\tau_{xx}^2(\alpha)\mathrm{w}^2(f-\alpha)d\alpha----(39)$$

$$\int_{-1/2}^{1/2}\tau_{xx}(\alpha)\tau_{xx}(-\alpha)w(f+\alpha)w(f-\alpha)\approx 0----(40)$$

A. Performance characteristics of this Method for PSE

Following equation shows that the value of estimate is asymptotically unbiased.[3]

$$Q_{BT} = 1.5\frac{N}{M}$$

$$\Delta f = \frac{1.28}{2M} = \frac{0.64}{M}$$

$$Q_{BT} = \frac{1.5}{0.64}N\Delta f = 2.34N\Delta f$$

We estimate the quality power for rectangular & triangular window.

V. COMPARISON OF THREE METHODS. (FOR QUALITY FACTOR) [1]

SR No.	Estimate	Quality Factor
1	Bartlett Method	$1.1N\Delta f$
2	Welch Method	$1.39N\Delta f$
3	Blackman & Tukey Method	$2.34N\Delta f$

A. Points Related to All Methods [1]

1. Due to windowing (leakage frequency due to side lobes) the frequency resolution is low in Bartlett method.
2. Welch Method has got better precision but less frequency resolution than Bartlett method.
3. Blackman-Tukey Method has better variance (even at large lags) and better precision than Bartlett & Welch Method methods. But frequency resolution is less than the Bartlett & Welch method

VI. CONCLUSION

1. Blackman-Tukey Method has better variance and better precision than Bartlett & Welch Method methods.
2. Bartlett Method has greater frequency resolution
3. Welch Method has better precision
4. Quality factor of Blackman & Tukey more than other two methods

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