Structure of Certain Ideals in Ternary Semirings

D. Madhusudana Rao, G. Srinivasa Rao

Abstract: In this paper we studied about principal ternary ideals, simple ternary ideals and semisimple ternary ideals in ternary semirings.

Mathematics Subject Classification: 16Y30, 16Y99.

Keywords: left simple, lateral simple, right simple, simple, duo ternary semiring, semisimple ternary semiring, globally idempotent.

I. INTRODUCTION


II. PRELIMINARIES

Definition II.1[13]: A nonempty set T together with a binary operation called addition and a ternary multiplication denoted by [ ] is said to be a ternary semiring if T is an additive commutative semigroup satisfying the following conditions:

i) \([abc]de = [a,bcd]e = [ab]cde\],
ii) \([a + b]c = [acd] + [bcd]\),
iii) \([a(b + c)] = [abd] + [acd]\),
iv) \([abc + d] = [abc] + [abd]\) for all \(a, b, c, d \in T\).

Throughout T will denote a ternary semiring unless otherwise stated.

Note II.2: For the convenience we write \(X_1X_2X_3\) instead of \([X_1X_2X_3]\).

Note II.3: Let T be a ternary semiring. If A,B and C are three subsets of T , we shall denote the set ABC = \(\{abc : a \in A, b \in B, c \in C\}\).

Note II.4: Let T be a ternary semiring. If A,B are two subsets of T , we shall denote the set \(A + B = \{a + b : a \in A, b \in B\}\).

Note II.5: Any semiring can be reduced to a ternary semiring.

Example II.6: Let T be an semigroup of all \(m \times n\) matrices over the set of all non negative rational numbers. Then T is a ternary semiring with matrix multiplication as the ternary operation.

Example II.7: Let \(S = \{\ldots, -2i, -i, 0, i, 2i, \ldots\}\) be a ternary semiring with respect to addition and complex triple multiplication.

Example II.8: The set T consisting of a single element 0 with binary operation defined by \(0 + 0 = 0\) and ternary operation defined by \(0.0.0 = 0\) is a ternary semiring. This ternary semiring is called the null ternary semiring or the zero ternary semirings.

Example II.9: The set Q of all rational numbers with respect to ordinary addition and ternary multiplication \([\ ]\) defined by \([abc] = abc\) for all \(a, b, c \in Q\) is a ternary semiring.

Definition II.10 [13]: A ternary semiring T is said to be commutative ternary semiring provided \(abc = bca = cba = bac = cba = acb\) for all \(a, b, c \in T\).

Example II.11: \((\mathbb{Z}^0, +, \cdot)\) is a ternary semiring of infinite order which is commutative.

Example II.12: The set \(2I\) of all evern integers is a commutative ternary semiring with respect to ordinary addition and ternary multiplication \([\ ]\) defined by \([abc] = abc\) for all \(a, b, c \in I\).

Definition II.13[12]: A ternary semigroup \((T, \ldots)\) is said to be left regular , if it satisfies the identity \(a = a\cdot xy \quad \forall a, x, y \in T\).

Definition II.14[12]: A ternary semigroup \((T, \ldots)\) is said to be right regular , if it satisfies the identity \(a = xay \quad \forall a, x, y \in T\).

Definition II.15 [12]: A ternary semigroup \((T, \ldots)\) is said to be lateral regular , if it satisfies the identity \(a = xa'y \quad \forall a, x, y \in T\).

Definition II.16 [12]: A ternary semigroup \((T, \ldots)\) is said to be two sided regular, if it left as well as right regular.

Definition II.17 [12]: A ternary semigroup \((T, \ldots)\) is said to be regular , it is left, lateral and right regular.

III. LEFT TERNARY IDEALS IN TERNARY SEMIRINGS

Definition III.1: A nonempty subset A of a ternary semiring T is said to be left ternary ideal or left ideal of T if

1) \(a, b \in A\) implies \(a + b \in A\).
2) \(b, c \in T, a \in A\) implies \(bca \in A\).
Note III.2: A nonempty subset A of a ternary semigroup T is a left ideal of T if and only if A is additive subsemigroup of T and T ∪ A ⊆ A.

Example III.3: In the ternary semiring \( \mathbb{Z}^0 \), \( n\mathbb{Z}^0 \) is a left ideal for any \( n \in \mathbb{N} \).

Theorem III.4: The nonempty intersection of any two left ideals of a ternary semiring T is a left ideal of T.

Theorem III.5: The nonempty intersection of any family of left ideals of a ternary semiring T is a left ideal of T.

Theorem III.6: The union of any two left ideals of a ternary semiring T is a left ideal of T.

Theorem III.7: The union of any family of left ideals of a ternary semiring T is a left ideal of T.

We now introduce a maximal left ternary ideal and left ternary semiring generated by a subset of a ternary semiring.

Definition III.8: An ideal A of a ternary semiring T is said to be a maximal left ternary ideal or simply maximal left ideal provided A is a proper left ideal of T and is not properly contained in any proper left ideal of T.

Definition III.9: Let T be a ternary semiring and A be a non-empty subset of T. The smallest left ideal of T containing A is called left ternary ideal of T generated by A.

Theorem III.10: The left ideal of a ternary semiring T generated by a non-empty subset A is the intersection of all left ideals of T containing A.

Proof: Let \( \Delta \) be the set of all left ideals of T containing A. Since T itself is a left ideal of T containing A, \( \Delta \subseteq T \). So \( \Delta \neq \emptyset \). Let \( S^* = \bigcap_{A \subseteq \Delta} S \). Since \( \Delta \subseteq S \) for all \( S \in \Delta \), \( \Delta \subseteq S^* \).

By theorem 3, 5, \( S^* \) is a left ideal of S. Let K be a left ideal of T containing A, K is a left ideal of T. Clearly \( A \subseteq K \). Therefore \( K \in \Delta \Rightarrow S^* \subseteq K \) and hence \( S^* \) is the left ideal of T generated by A.

We now introduce a principal left ideal of a ternary semiring and characterize principal left ideal.

Definition III.11: A left ideal A of a ternary semiring T is said to be the principal left ternary ideal generated by \( a \) if A is a left ternary ideal generated by \( \{a\} \) for some \( a \in T \).

It is denoted by \( L(a) \) or \( < a > \).

Theorem III.12: If T is a ternary semiring and \( a \in T \) then \( L(a) = \left\{ \sum_{i=1}^{n} r_i t_i a + n a : r_i, t_i \in T, n \in \mathbb{Z}^+ \right\} \).

Where \( \Sigma \) denotes a finite sum and \( \mathbb{Z}^+ \) is the set of all positive integers with zero.

Proof: Let \( A = \left\{ \sum_{i=1}^{n} r_i t_i a + n a : r_i, t_i \in T, n \in \mathbb{Z}^+ \right\} \). Let \( a, b \in A \).

\( a, b \in A \). Then \( a = \sum_{i=1}^{n} r_i t_i a + n a \) and \( b = \sum_{i=1}^{n} r_j t_j a + n a \) for \( r_i, t_i \in T, r_j, t_j \in T, n \in \mathbb{Z}^+ \).

Now \( a + b = \sum_{i=1}^{n} r_i t_i a + n a + \sum_{i=1}^{n} r_j t_j a + n a \Rightarrow a + b \) is a finite sum.

Therefore \( a + b \in A \) and hence A is an additive subsemigroup of T.

For \( r_1, a \in T \) and \( a \in A \).

Then \( t_{12} t_2 a = t_{12} t_2 (\sum_{i=1}^{n} r_i t_i a + n a) = \sum_{i=1}^{n} r_i t_i (t_{12} t_2 a) + n (t_{12} t_2 a) \in A \)

Therefore \( t_{12} t_2 a \in A \) and hence A is a left ideal of T.

Let L be a left ideal of T containing a.

Let \( r \in A \). Then \( r = \sum_{i=1}^{n} r_i t_i a + n a \) for \( r_i, t_i \in T, n \in \mathbb{Z}^+ \).

If \( r = \sum_{i=1}^{n} r_i t_i a + n a \in L \). Therefore \( A \subseteq L \) and hence A is a smallest ideal containing a.

Therefore A = \( L(a) = \left\{ \sum_{i=1}^{n} r_i t_i a + n a : r_i, t_i \in T, n \in \mathbb{Z}^+ \right\} \).

Note III.13: If T is ternary semiring and \( a \in T \) then \( L(a) = T \cdot T \cdot a + a \cdot a \).

We now introduce a left simple ternary semigroup and characterize left simple ternary semigroups.

Definition III.14: A ternary semiring T is said to be left simple ternary semiring if T is its only left ternary ideal.

Theorem III.15: A ternary semiring T is a left simple ternary semiring if and only if \( TTa = T \) for all \( a \in T \).

Proof: Suppose that T is a left simple ternary semiring and \( a \in T \). Let \( s, v \in TTa \); \( t, u \in T \).

\( s, v \in TTa \Rightarrow s = \sum_{i=1}^{n} v_i w_i a \)

\( v = \sum_{i=1}^{n} v_j w_j a \) where \( v_i, v_j, w_i, w_j \in T \), \( n \in \mathbb{Z}^+ \).

\( s + v = \sum_{i=1}^{n} v_i w_i a + \sum_{j=1}^{n} v_j w_j a \) is a finite sum. Therefore \( s + v \in TTa \) and hence TTa is a subsemigroup of \( (T, +) \).

Now \( uTSu = uT (\sum_{i=1}^{n} v_i w_i a) \)

\( = (\sum_{i=1}^{n} u v_i w_i a) \in TTa \Rightarrow TTa \) is a left ideal of T. Since T is a left simple ternary semigroup, \( TTa = T \).

Therefore \( TTa = T \) for all \( a \in T \).

Conversely suppose that \( TTa = T \) for all \( a \in T \).

Let L be a left ideal of T. Let \( l \in L \). Then \( l \in T \). By assumption TTl = T.

Let \( t \in T \). Then \( t \in TTl \).
\[ t = \sum_{i=1}^{n} u_i v_i \text{ for some } u_i, v_i \in T. \]

\[ l \in L; u_k, v_i \in T \text{ and } L \text{ is a left ideal of } T \Rightarrow \sum_{i=1}^{n} u_i v_i \in L. \]

Therefore \( T \subseteq L \). Clearly \( L \subseteq T \) and hence \( L = T \).

Therefore \( T \) is the only left ideal of \( T \). Hence \( T \) is left simple ternary semiring.

IV. LATERAL TERNARY IDEALS

We now introduce the term of a lateral ternary ideal in a ternary semiring.

**Definition IV.1:** A nonempty subset of a ternary semiring \( T \) is said to be a *lateral ternary ideal* or simply *lateral ideal* of \( T \) if

1. \( a, b \in A \) implies \( a + b \in A \).
2. \( b, c \in T \), \( a \in A \) implies \( abc \in A \).

**Example IV.3:** In the ternary semiring \( Z \), \( nZ \) is a lateral ideal for any \( n \in N \).

**Theorem IV.4:** The nonempty intersection of any two lateral ideals of a ternary semiring \( T \) is a lateral ideal of \( T \).

**Theorem IV.5:** The nonempty intersection of any family of lateral ideals of a ternary semiring \( T \) is a lateral ideal of \( T \).

**Theorem IV.6:** The union of any two lateral ideals of a ternary semiring \( T \) is a lateral ideal of \( T \).

**Theorem IV.7:** The union of any family of lateral ideals of a ternary semiring \( T \) is a lateral ideal of \( T \).

We now introduce a maximal lateral ternary ideal and lateral ternary ideal generated by a subset of a ternary semiring.

**Definition IV.8:** An ideal \( A \) of a ternary semiring \( T \) is said to be a *maximal lateral ternary ideal* provided \( A \) is a proper lateral ideal of \( T \) and is not properly contained in any proper lateral ideal of \( T \).

**Definition IV.9:** Let \( T \) be a ternary semiring and \( A \) be a non-empty subset of \( T \). The smallest lateral ideal of \( T \) containing \( A \) is called *lateral ternary ideal of \( T \) generated by \( A \).*

**Theorem IV.10:** The lateral ideal of a ternary semiring \( T \) generated by a non-empty subset \( A \) is the intersection of all lateral ideals of \( T \) containing \( A \).

We now introduce a principal lateral ternary ideal of a ternary semiring and characterize principal lateral ternary ideal.

**Definition IV.11:** A lateral ideal \( A \) of a ternary semiring \( T \) is said to be the *principal lateral ternary ideal generated by \( a \)* if \( A \) is a lateral ideal generated by \( \{a\} \) for some \( a \in T \). It is denoted by \( M(a) \) (or) \( \langle a \rangle_\text{m} \).

**Theorem IV.12:** If \( T \) is a ternary semiring and \( a \in T \) then

\[ M(a) = \left\{ \sum_{i=1}^{r} r_i a_i + \sum_{j=1}^{n} u_j v_j p_j q_j + na \mid r_i, u_j, v_j, p_j, q_j \in T, n \in z_{n}^+ \right\}. \]

\( \Sigma \) denotes a finite sum and \( z_{n}^+ \) is the set of all positive integer with zero.

**Proof:** Similar to proof of the theorem IV.12.

**Note IV.13:** If \( T \) is ternary semiring and \( a \in T \) then

\[ L(a) = T^+ aT^+ + T^+ T^+ aT^+ + na. \]

We now introduce a lateral simple ternary semiring and characterize lateral simple ternary semiring.

**Definition IV.14:** A ternary semiring \( T \) is said to be *lateral simple ternary semiring* if \( T \) is its only lateral ideal.

**Theorem IV.15:** A ternary semiring \( T \) is a lateral simple ternary semiring if and only if \( T = T = TA = TT = T \) for all \( a \in T \).

**Proof:** Similar to III.15.

V. RIGHT TERNARY IDEALS

We now introduce the term of a right ternary ideal in a ternary semiring.

**Definition V.1:** A nonempty subset \( A \) of a ternary semigroup \( T \) is a *right ternary ideal* or simply *right ideal* of \( T \) if

1. \( a, b \in A \) implies \( a + b \in A \).
2. \( b, c \in T \), \( a \in A \) implies \( abc \in A \).

**Note V.2:** A nonempty subset \( A \) of a ternary semigroup \( T \) is a right ideal of \( T \) if and only if \( A \) is additive subsemigroup of \( T \) and \( AT \subseteq A \).

**Example V.3:** In the ternary semiring \( Z \), \( nZ \) is a right ideal for any \( n \in N \).

**Theorem V.4:** The nonempty intersection of any two right ideals of a ternary semiring \( T \) is a right ideal of \( T \).

**Theorem V.5:** The nonempty intersection of any family of right ideals of a ternary semiring \( T \) is a right ideal of \( T \).

**Theorem V.6:** The union of any two right ideals of a ternary semiring \( T \) is a right ideal of \( T \).

**Theorem V.7:** The union of any family of right ideals of a ternary semiring \( T \) is a right ideal of \( T \).

We now introduce a maximal right ideal and right ideal generated by a subset of a ternary semiring.

**Definition V.8:** An ideal \( A \) of a ternary semiring \( T \) is said to be a *maximal right ideal* provided \( A \) is a proper right ideal of \( T \) and is not properly contained in any proper right ideal of \( T \).

**Definition V.9:** Let \( T \) be a ternary semiring and \( A \) be a non-empty subset of \( T \). The smallest right ideal of \( T \) containing \( A \) is called *right ternary ideal of \( T \) generated by \( A \).*

**Theorem V.10:** The right ideal of a ternary semigroup \( T \) generated by a non-empty subset \( A \) is the intersection of all right ideals of \( T \) containing \( A \).

We now introduce a principal right ideal of a ternary semiring and characterize principal right ideal.
Structure of Certain Ideals in Ternary Semirings

**Definition VI.11**: A right ideal $A$ of a ternary semiring $T$ is said to be a principal right ideal generated by $a$ if $A$ is a right ideal generated by $\{a\}$ for some $a \in T$. It is denoted by $R(a)$ or $<a>$. 

**Theorem V.12**: If $T$ is a ternary semiring and $a \in T$ then $R(a) = \{ \sum_{i=1}^{n} ar_t_i + na : r_i, t_i \in T, n \in \mathbb{Z}_0^+ \}$. Where $\Sigma$ denotes a finite sum and $\mathbb{Z}_0^+$ is the set of all positive integer with zero. 

**Proof**: Similar to the proof of the theorem III.12.

**Note VI.13**: If $T$ is a ternary semiring and $a \in T$ then $R(a) = aT + na$.

We now introduce a right simple ternary semiring and characterize right simple ternary semiring.

**Definition VI.14**: A ternary semiring $T$ is said to be right simple ternary semiring if $T$ is its only right ideal.

**Theorem VI.15**: A ternary semiring $T$ is a right simple ternary semiring if and only if $aT = T$ for all $a \in T$.

**Proof**: Similar to III.15.

---

**VI. TWO SIDED TERNARY IDEALS**

We now introduce the term of a two sided ternary ideal in a ternary semiring.

**Definition VI.1**: A nonempty subset $A$ of a ternary semiring $T$ is a two sided ternary ideal or simply two sided ideal of $T$ if

1. $a, b \in A$ implies $a + b \in A$
2. $b, c \in T, a \in A$ implies $bca \in A, abc \in A$.

**Note VI.2**: A nonempty subset $A$ of a ternary semiring $T$ is a two sided ideal of $T$ if and only if $A$ is both a left ideal and a right ideal of $T$.

**Example VI.3**: In the ternary semiring $\mathbb{Z}^0$, $n\mathbb{Z}^0$ is a two sided ideal for any $n \in \mathbb{N}$.

**Theorem VI.4**: The nonempty intersection of any two two sided ideals of a ternary semiring $T$ is a two sided ideal of $T$.

**Theorem VI.5**: The nonempty intersection of any family of two sided ideals of a ternary semiring $T$ is a two sided ideal of $T$.

**Theorem VI.6**: The union of any two two sided ideals of a ternary semiring $T$ is a two sided ideal of $T$.

**Theorem VI.7**: The union of any family of two sided ideals of a ternary semiring $T$ is a two sided ideal of $T$.

We now introduce a maximal two sided ideal and two sided ideal generated by a subset of a ternary semiring.

**Definition VI.8**: An ideal $A$ of a ternary semiring $T$ is said to be a maximal two sided ideal provided $A$ is a proper two sided ideal of $T$ and is not properly contained in any proper two sided ideal of $T$.

**Definition VI.9**: Let $T$ be a ternary semiring and $A$ be a non-empty subset of $T$. The smallest two sided ideal of $T$ containing $A$ is called two sided ideal of $T$ generated by $A$.

**Theorem VI.10**: The two sided ideal of a ternary semiring $T$ generated by a non-empty subset $A$ is the intersection of all two sided ideals of $T$ containing $A$.

We now introduce a principal two sided ideal of a ternary semiring $T$ and characterize principal two sided ideal.

**Definition VI.11**: A two sided ideal $A$ of a ternary semiring $T$ is said to be the principal two sided ideal provided $A$ is a two sided ideal generated by $\{a\}$ for some $a \in T$. It is denoted by $D(a)$ or $<a>$. 

**Theorem VI.12**: If $T$ is a ternary semigroup and $a \in T$ then

$$
T(a) = \left\{ \sum_{i=1}^{n} r_i s_i a + \sum_{j=1}^{m} u_j + \sum_{k=1}^{l} m_k p_k q_k : r_i, s_i, t_i, u_j, l_k \in T, n, m, l \in \mathbb{Z}_0^+ \right\}.
$$

$\Sigma$ denotes a finite sum and $\mathbb{Z}_0^+$ is the set of all positive integer with zero.

**Proof**: Similar to the proof of the theorem III.12.

**Note VI.13**: If $T$ is a ternary semiring and $a \in T$ then $T(a) = T a + T a + T a + T a + T a + T a + T a + T a + T a + T a + T a$.

**Definition VI.14**: A ternary semiring $T$ is said to be a left duo ternary semiring provided every left ideal of $T$ is a two sided ideal of $T$.

**Definition VI.15**: A ternary semiring $T$ is said to be a right duo ternary semiring provided every right ideal of $T$ is a two sided ideal of $T$.

**Definition VI.16**: A ternary semiring $T$ is said to be a duo ternary semiring provided it is both a left duo ternary semiring and a right duo ternary semiring.

**Theorem VI.17**: A ternary semiring $T$ is a duo ternary semiring if and only if $xT^T = T^T x$ for all $x \in T$.

**Proof**: Suppose that $T$ is a duo ternary semiring and $x \in T$.

Let $t \in xT^T$. Then $t = \sum_{i=1}^{n} x_i u_i v_i$ for some $u_i, v_i \in T^T$.

Since $T^T x$ is a left ideal of $T$, $T^T x$ is a ideal of $T$.

So $x \in T^T x, u_i, v_i \in T, T^T x$ is a ideal of $T$.

Therefore $xT^T \subseteq T^T x$. Similarly we can prove that $T^T x \subseteq xT^T$.

Therefore $xT^T = T^T x$ for all $x \in T$. Conversely suppose that $xT^T = T^T x$ for all $x \in T$. Let $A$ be a left ideal of $T$.

Let $x \in A, u_i, v_i \in T$.

Then

$$
\sum_{i=1}^{n} x_i u_i v_i \in xT^T = T^T x
$$

Therefore $xT^T \subseteq T^T x$. Similarly we can prove that $T^T x \subseteq xT^T$.

Therefore $xT^T = T^T x$ for all $x \in T$.

Conversely suppose that $xT^T = T^T x$ for all $x \in T$. Let $A$ be a left ideal of $T$.

Let $x \in A, s_i, t_i \in T, A$ is a left ideal of $T$.

Therefore $xT^T = T^T x$ for all $x \in T$.

Let $x \in A, s_i, t_i \in T, A$ is a left ideal of $T$.

Then

$$
\sum_{i=1}^{n} x_i u_i v_i = \sum_{i=1}^{n} s_i f_i x
$$

for some $s_i, t_i \in T$.

Let $x \in A, s_i, t_i \in T, A$ is a left ideal of $T$.

Therefore $xT^T = T^T x$ for all $x \in T$.

Conversely suppose that $xT^T = T^T x$ for all $x \in T$. Let $A$ be a left ideal of $T$.

Therefore $xT^T = T^T x$ for all $x \in T$.
Therefore A is a right ideal of T and hence A is an ideal of T. Therefore T is left duo ternary semiring. Similarly we can prove that T is a right duo ternary semiring. Hence T is duo ternary semiring.

**Theorem VI.18:** Every commutative ternary semiring is a duo ternary semiring.

**Proof:** Suppose that T is a commutative ternary semiring. Therefore
\[ \forall x, y \in T, \quad xTT = TTx \quad \forall x \in T. \]
By theorem VI.17, T is a duo ternary semiring.

**Theorem VI.19:** Every normal ternary semiring is a duo ternary semiring.

**Proof:** Suppose that T is normal ternary semiring. Then \( xyT = Txy \) for all \( x, y \in T \)
\[ \Rightarrow xTT = TTx \quad \forall x \in T \]
Therefore by theorem VI.17, T is a duo ternary semiring.

**Theorem VI.20:** Every quasi commutative ternary semiring is a duo ternary semiring.

**Proof:** Suppose that T is a quasi commutative ternary semiring. Then for each \( a, b, c \in T \), there exists a natural number \( n \) such that \( abc = b^n ac = bca = c^n ba = cab \). Let A be a left ideal of T. Therefore \( TTA \subseteq A \).

Note VII.2: A nonempty subset A of a ternary semigroup T is an ideal of T if and only if it is left ideal, lateral ideal and right ideal of T.

**Example VII.3:** Let N be the set of all natural numbers. Define the ternary operation from \( \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) as \( (a, b, c) = a \cdot b \cdot c \) where \( \cdot \) is usual multiplication. Then N is a ternary semiring and \( A = 3\mathbb{N} \) is an ideal of the ternary semiring N.

**Example VII.4:** Consider the ternary semiring \( \mathbb{Z}_0^+ \) under usual addition and ternary multiplication. Let \( I = \{0, -1\} \cup \{-5, -6, -7, \ldots\} \) is an ideal of \( \mathbb{Z}_0^+ \).

**Theorem VII.5:** The nonempty intersection of any two ideals of a ternary semiring T is an ideal of T.

**Theorem VII.6:** The nonempty intersection of any family of ideals of a ternary semiring T is an ideal of T.

**Theorem VII.7:** The union of any two ideals of a ternary semiring T is a ternary ideal of T.

**Theorem VII.8:** The union of any family of ideals of a ternary semiring T is an ideal of T. We now introduce a proper ideal, trivial ideal, maximal ideal and globally idempotent ideal of a ternary semiring and globally idempotent ternary semiring.

**Definition VII.9:** An ideal A of a ternary semiring T is said to be a **proper ideal** of T if A is different from T.

**Definition VII.10:** An ideal A of a ternary semiring T is said to be a **trivial ideal** provided TA is singleton.

**Definition VII.11:** An ideal A of a ternary semiring T is said to be a **maximal ideal** provided A is a proper ideal of T and is not properly contained in any proper ideal of T.

**Definition VII.12:** An ideal A of a ternary semiring T is called a **globally idempotent ideal** if \( A^n = A \) for all odd natural number n.

**Definition VII.13:** A ternary semigroup T is said to be a **globally idempotent ternary semiring** if \( TT^T = T \) for all odd natural number n.

**Theorem VII.14:** If A is an ideal of a ternary semiring T with multiplicative identity \( e \) and \( e \in A \) then \( A = T \).

**Proof:** Clearly A \( \subseteq T \). Let \( t \in T \), \( t \in A \). Since T is a ternary semiring
\[ \Rightarrow T \subseteq A \quad A \subseteq T \quad T \subseteq A \Rightarrow T = A. \]

**Theorem VII.15:** If T is a ternary semiring with multiplicative identity e then the union of all proper ideals of T is the unique maximal ideal of T.

**Proof:** Let M be the union of all proper ideals of T. Since e is not an element of any proper ideal of T, e \( \notin M \). Therefore M is a proper subset of T. By theorem 7.8, M is a ternary ideal of T. Thus M is a proper ideal of T. Since M contains all proper ideals of T, M is a maximal ideal of T. If \( M_1 \) is any maximal ideal of T, then \( M_1 \subseteq M \subseteq T \) and hence \( M_1 = M \). Therefore M is the unique maximal ideal of T.

We now introducing ideal generated by a subset of a ternary semiring.

**Definition VII.16:** Let T be a ternary semiring and \( A \) be a non-empty subset of T. The smallest ideal of T containing \( A \) is called **ideal of T generated by A**.

**Theorem VII.17:** The ideal of a ternary semiring T generated by a non-empty subset A is the intersection of all ideals of T containing A.

We now introduce a principal ideal and characterize principal ideal of a ternary semiring.

**Definition VII.18:** An ideal A of a ternary semiring T is said to be a **principal ideal** provided A is an ideal generated by \( \{a\} \) for some \( a \in T \). It is denoted by \( J(a) \) or \( (a) < a > \).

**Theorem VII.19:** If T is a ternary semiring and \( a \in T \) then \( J(a) = \{ \sum_{i=1}^{n} p_i q_i a + \sum_{j=1}^{m} a r_j s_j + \sum_{k=1}^{n} t_k u_k + \sum_{l=1}^{n} v_l w_l x_l y_l + na \}
\[ : p_i, q_i, r_j, s_j, t_k, u_k, v_l, w_l, x_l, y_l \in T, n \in \mathbb{Z}_0^+ \} \]
Where $\Sigma$ denotes a finite sum and $z_0^+$ is the set of all positive integer with zero.

**Proof:** Let

$$A = \sum_{l=1}^n v_jw_i a_{ij} y_i + na$$

$$B = \sum_{k=1}^m \sum_{g=1}^g l_k a_{kg} u_k + b_{kg}$$

Then

$$A + B = \sum_{l=1}^n p_lq_l a + \sum_{j=1}^n ar_j s_j + \sum_{k=1}^m t_k a_{lk} u_k + \sum_{h=1}^h v_j w_i a_{ij} y_i + na$$

Note VII.20: If $T$ is a ternary semigroup and $a \in T$ then $J(a) = a + TTa + aTT + TaT + TTaTT = T^{'a}T^{'a}T^{'a}$.

**Theorem VII.21:** Let $A, B$ be two ideals of ternary semiring $T$ with zero and let $A + B = \{ a + b : a \in A, b \in B \}$. Then $A + B$ is an ideal generated by $(A \cup B)$.

**Proof:** We first show that $A + B$ is an ideal. Let $a_1 + b_1 \in A + B$ and $a_2 + b_2 \in A + B$.

Let $s, t$ be any arbitrary elements of $T$. Then, since $A, B$ are ideals.

Then we have $a_1, a_2 \in A, b_1, b_2 \in B$

$\Rightarrow a_1 + a_2 \in A$ and $b_1 + b_2 \in B$

$\Rightarrow (a_1 + a_2) + (b_1 + b_2) \in A + B \Rightarrow (a_1 + b_1) + (a_2 + b_2) \in A + B$

Now $a_1, a_2 \in A, b_1, b_2 \in B, s, t \in T$, we have $sta_1, sta_1, a_1st \in A$

$sdb_1, sb_1t, b_1st \in B$

Therefore $A + B$ is an ideal of $T$. We now show that $A + B$ is generated by $A \cup B$.

Let $a \in A$. Since $0 \in B$, we have $a + 0 \in A + B$. But $a + 0 = a$.

Thus $a \in A \Rightarrow a \in A + B$.

Therefore $A \subseteq A + B$. Similarly $B \subseteq A + B$. It follows that $A \cup B \subseteq A + B$. Thus $A + B$ is an ideal containing $A \cup B$.

Also if any ideal contains $A \subset B$, then it is easy to see that it must contain $A + B$. Hence $A + B$ is the smallest ideal containing $A \cup B$, i.e., $A + B$ is the ideal generated by $A \cup B$.

In other words, $A + B = (A \cup B)$.

**Corollary VII.22:** The left ideal generated by the union $A \cup B$ of two left ideals is the set $A + B$ consisting of the elements of ternary semiring $T$ obtained on adding any element of $A$ to any element of $B$.

**Theorem VII.23:** In any ternary semiring $T$, the following are equivalent.

1) Principal ideals of $T$ form a chain.

2) Ideals of $T$ form a chain.

**Proof:** (1) $\Rightarrow$ (2): Suppose that principal ideals of $T$ form a chain.

Let $A, B$ be two ideals of $T$. Suppose if possible $A \nsubseteq B, B \nsubseteq A$.

Then there exist $a \in AB$ and $b \in BA$

$a \in A \Rightarrow \not{\circ} a \nsubseteq A$ and $b \in B \Rightarrow \not{\circ} b \nsubseteq B$.

Since principal ideals form a chain, either $\not{\circ} a \subseteq \not{\circ} b$ or $\not{\circ} b \subseteq \not{\circ} a$.

If $\not{\circ} a \subseteq \not{\circ} b$, then $a \in b \subseteq B$. It is a contradiction.

If $\not{\circ} b \subseteq \not{\circ} a$, then $b \in a \subseteq A$. It is also a contradiction.
Therefore $A \subseteq B$ or $B \subseteq A$ and hence ideals form a chain.

(2) $\implies$ (1) : Suppose that ideals of $T$ form a chain.

Then clearly principal ideals of $T$ form a chain.

We now introduce a simple ternary semiring and characterize simple ternary semirings.

**Definition VII.24:** A ternary semigroup $T$ is said to be a simple ternary semiring if $T$ is its only ideal of $T$.

**Theorem VII.25:** If $T$ is a left simple ternary semiring or a lateral simple ternary semiring or a right simple ternary semiring then $T$ is a simple ternary semiring.

**Proof:** Suppose that $T$ is a left simple ternary semiring. Then $T$ is the only left ideal of $T$. If $A$ is an ideal of $T$, then $A$ is a left ideal of $T$ and hence $A = T$.

Therefore $T$ itself is the only ideal of $T$ and hence $T$ is a simple ternary semiring.

Suppose that $T$ is a lateral simple ternary semiring. Then $T$ is the only lateral ideal of $T$. If $A$ is an ideal of $T$, then $A$ is a lateral ideal of $T$ and hence $A = T$.

Therefore $T$ itself is the only ideal of $T$ and hence $T$ is a simple ternary semiring. Similarly if $T$ is right simple ternary semiring then $T$ is simple ternary semiring.

**Theorem VII.26:** A ternary semiring $T$ is simple ternary semiring if and only if $\text{TT}aTT = T$ for all $a \in T$.

**Proof:** Suppose that $T$ is a simple ternary semiring and $a \in T$. Let $s, t \in \text{TT}aTT; \ t, u \in T$

$s, t \in \text{TT}aTT \implies s = \sum_{j=1}^{n} r_j s_j a v_j w_j$ and $t = \sum_{j=1}^{n} r_j s_j a v_j w_j$

where $r_j, r_j, s_j, s_j, v_j, v_j, w_j, w_j \in T$,

$n \in \mathbb{Z}^{+}$. Therefore

$s + t = \sum_{j=1}^{n} r_j s_j a v_j w_j + \sum_{j=1}^{n} r_j s_j a v_j w_j$

is a finite sum and hence $s + t \in \text{TT}aTT$. Therefore $\text{TT}aTT$ is a subsemigroup of $(T, +)$.

Now $sut = (\sum_{j=1}^{n} r_j s_j a v_j w_j)ut$

$= (\sum_{j=1}^{n} r_j s_j a u v_j w_j) \in \text{TT}aTT$ and

$uts = ut(\sum_{j=1}^{n} r_j s_j a v_j w_j)$

$= \sum_{j=1}^{n} u t r_j s_j a v_j w_j \in \text{TT}aTT$

$\implies \text{TT}aTT$ is a ideal of $T$. Since $T$ is a simple ternary semiring, $\text{TT}aTT = T$

Therefore $\text{TT}aTT = T$ for all $a \in T$.

Conversely suppose that $\text{TT}aTT = T$ for all $a \in T$.

Let $I$ be an ideal of $T$. Let $l \in I$. Then $l \in T$ . By assumption $\text{TT}I = T$.

Let $t \in T$. Then $t \in \text{TT}IT$

$\implies t = \sum_{j=1}^{n} r_j s_j a v_j w_j$ for some $r_j, s_j, u_j, v_j \in T$ and $l$ is an ideal of $T$

$\implies \sum_{j=1}^{n} r_j s_j a v_j w_j \in T \implies t \in R$.

Therefore $T \subseteq I$. Clearly $I \subseteq T$ and hence $I = T$. Therefore $T$ is the only ideal of $T$. Hence $T$ is a simple ternary semiring.

**Theorem VII.27:** A ternary semiring $T$ is regular then every principal ideal of $T$ is generated by an idempotent.

**Proof:** Suppose $T$ is a regular ternary semiring. Let $a > b$ be a principal ideal of $T$. Since $T$ is regular, $\exists x, y \in T \exists axay = a$.

Now $(axay)^3 = axayaxayax = axayaxay = axay$. Let $axay = e$.

$a = axay = axayaxay = eea \in < e > \implies a > e \in < e >$.

Now $e = axay \in < a > \implies a > e \in < a >$. Therefore $a > e$ and hence every principal ideal generated by an idempotent.

We now introduce a semi simple element of a ternary semigroup and a semi simple ternary semiring.

**Definition VII.28:** An element $a$ of a ternary semiring $T$ is said to be semi simple if $a < a >^3$ i.e. $a >^3 = < a >$.

**Theorem VII.29:** An element $a$ of a ternary semiring $T$ is said to be semi simple if $a < a >^n$ i.e. $a >^n = < a >$ for all odd natural number $n$.

**Proof:** Suppose that $a$ is semi simple element of $T$. Then $a >^3 = < a >$.

Let $a \in T$ and $n$ is odd natural number.

If $n = 1$ then clearly $a < a >$.

If $n = 3$ and $a$ is semi simple then $a >^3 = < a >$.

If $n = 5$ then $a >^3 = < a >^3 < a >^3 = < a >$.

Therefore by induction of $n$ is an odd natural number, we have $a >^n = < a >$.

The converse part is trivial.

**Definition VII.30:** A ternary semiring $T$ is called semi simple ternary semiring provided every element in $T$ is semi simple.

**Theorem VII.31:** Let $T$ be a ternary semiring and $a \in T$.

If $a$ is regular then $a$ is semi simple.

**Proof:** Suppose that $a$ is regular. Then $a = axay$ for some $x, y \in T \implies a \in < a >^3$.

Therefore $a$ is semi simple.

**Theorem VII.32:** Let $a$ be an element of a ternary semiring $T$. If $a$ is left regular or lateral regular or right regular, then $a$ is semi simple.

**Proof:** Suppose $a$ is left regular. Then $a = axa$ for some $x, y \in S \implies a \in < a >^3$.

Therefore $a$ is semi simple.

If $a$ is right regular, then $a = yax$ for some $x, y \in S \implies a \in < a >^3$.

Therefore $a$ is semi simple.

**Theorem VII.33:** Let $a$ be an element of a ternary semiring $T$. If $a$ is intraregular then $a$ is semi simple.
Structure of Certain Ideals in Ternary Semirings

Proof: Suppose $a$ is intra regular. Then $a = xa^2y = xa^2a'y$ for some $x, y \in T$ 
$\Rightarrow a \in a >^2$. Therefore $a$ is semi simple.

VIII. CONCLUSION

In this paper mainly we studied about structure of certain ideals in ternary semirings.

ACKNOWLEDGMENT

The authors would like to thank the referee(s) for careful reading of the manuscript.

REFERENCES


AUTHORS PROFILE

Dr. D. Madhusudhana Rao, completed his M.Sc. from Osmania University, Hyderabad, Telangana, India. M. Phil. from M. K. University, Madurai, Tamil Nadu, India. Ph. D. from Acharya Nagarjuna University, Andhra Pradesh, India. He joined as Lecturer in Mathematics, in the department of Mathematics, VSR & NVR College, Tenali, A. P. India in the year 1997, after that he promoted as Head, Department of Mathematics, VSR & NVR College, Tenali. He helped more than 5 Ph.D’s and at present he guided 5 Ph. D. Scalars and 3 M. Phil., Scalars in the department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar, Guntur, A. P. A major part of his research work has been devoted to the use of semigroups, Gamma semigroups, duo gamma semigroups, partially ordered gamma semigroups and ternary semigroups, Gamma semirings and ternary semirings. Near rings etc. He acting as peer review member to the “British Journal of Mathematics & Computer Science”. He published more than 33 research papers in different International Journals in the last two academic years.

G. Srinivasa Rao, He is working as an Assistant Professor in the Department of Applied Sciences & Humanities, Tirumala Engineering College. He completed his M.Phil. in Madhurai Kamaraj University. He is pursuing Ph.D. under the guidance of Dr.D.Madhusudanarao in Acharya Nagarjuna University. He published more than 3 research papers in popular International Journals to his credit. His area of interests are ternary semirings, ordered ternary semirings, semirings and topology. Presently he is working on Ternary semirings.