General Pattern of Total Coloring of a Prism Graph of $n$-Layers and a Grid Graph

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Abstract: Behzad [1] introduced the total coloring of a graph $G$ as an assignment of colors to the elements (vertices and edges) of $G$ such that no two elements receive the same color. In this paper, we have obtained the total coloring of prism graph of $n$-layers and grid graphs in general using five colors only and found that the total-chromatic number is $\Delta(G) + 1 = 5$ for both prism graph of $n$-layers and grid graphs. AMS Subject Classification: 05C15

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I. INTRODUCTION

Behzad [1] has defined the total coloring of a graph. A graph $G$ is said to admit total coloring if
(i) no two adjacent vertices have the same color
(ii) no two adjacent edges have the same color
(iii) no edge and its end vertices are assigned with the same color.

The minimum number of colors required for a graph $G$ in this way, is its chromatic number and it is denoted by $\chi_t(G)$. Behzad [1] discussed about the total coloring of paths, cycles, complete graphs and complete bipartite graphs. Behzad [1] and Vizing [2] have given the total coloring conjecture as $\Delta(G) + 1 \leq \chi_t(G) \leq \Delta(G) + 2$ for a simple graph $G$. Jensen et.al [3] and Borodin [4] have also discussed about the total coloring of graphs. Gutman [5] gave the total coloring of thorny graphs. Heckman et.al [6] discussed the circular total coloring of graphs. Sudha et.al [7,8] have given the total coloring and $(k,d)$-total coloring for prisms $Y_n$ and $S(n,m)$-graphs (Sudha graphs). The prism graph $Y_n$ is the cartesian product of the cycle $C_m$ with the path $P_n$.

DEFINITION 1: A prism graph of $n$-layers, $Y^n_m$ is a simple graph given by the cartesian product of the cycle $C_m$ and the path $P_n$. This graph consists of $mn$ vertices and $m(2n-1)$ edges. The number of circles in $Y^n_m$ is $n$ which depends on the length of path, $P_n$ and we say that the prism is of $n$-layers.

ILLUSTRATION 1: Consider the prism graph $C_4 \times P_3$. Let the vertex sets of the cycle $C_4$ and the path $P_3$ be $\{u_1,u_2,u_3,u_4\}$ and $\{w_1,w_2,w_3,w_4\}$ respectively. By the above definition, we obtain the prism graph of 3-layers with the vertices denoted by $v_{i,j}$ for $i = 1,2,3$ and $j = 1,2,3,4$ respectively.

DEFINITION 2: A grid graph is defined as the cartesian product of two paths $P_m$ and $P_n$.

The graph $P_m \times P_n$ has vertices denoted by $v_{i,j}$ and the number of edges in $P_m \times P_n$ is $2mn-n$.

The vertex set and the edge set of $P_m \times P_n$ are given as $V(P_m \times P_n) = \{(g,h) | g \in V(P_m) \text{ and } h \in V(P_n)\}$ and $E(P_m \times P_n) = \{\{(g,h) | g \in V(P_m) \text{ and } h \in V(P_n)\} | g \text{ and } h \in E(P_m) \text{ and } g \in E(P_n)\}$.

ILLUSTRATION 2: Consider the grid graph $P_3 \times P_4$. Let the vertex sets of the paths $P_3$ and $P_4$ be $\{u_1,u_2,u_3\}$ and $\{w_1,w_2,w_3,w_4\}$ respectively. By the above definition of grid graphs, we obtain the cartesian product $P_3 \times P_4$ with the vertices denoted by $v_{i,j}$ for $i = 1,2,3$ and $j = 1,2,3,4$ respectively.

In this paper, we have assigned the colors for both vertices and edges of the prism graph of $n$-layers as well as grid graphs in such a way that it satisfies the definition of total coloring and its total-chromatic number is found to be $\Delta(G) + 1 = 5$ for both the graphs.

II. TOTAL COLORING OF PRISM GRAPH OF $n$-LAYERS

Theorem 1. The prism graph of $n$-layers, $Y^n_m$ ($m \geq 3$, $n \geq 3$), admits total coloring and its total-chromatic number is 5.

Proof: Consider the prism graph $Y^n_m$ ($m \geq 3$, $n \geq 3$), which has $n$-layers with $m$ vertices on each layer.

Let the vertex set of the prism graph of $n$-layers $Y^n_m$ is given by $V(Y^n_m) = \bigcup_{i=1}^{n} \{v_{i,j} \mid 1 \leq j \leq m\}$.
General Pattern of Total Coloring of a Prism Graph of \(n\)-Layers and a Grid Graph

where \(i\) stands for the layer and \(j\) is its vertex on the \(i^{th}\) layer.
and its edge set is given by
\[
E(Y^n_m) = \bigcup_{i=1}^{n} (v_{ij}v_{i,j+1}; 1 \leq j \leq m) \\
\bigcup_{i=1}^{n} (v_{ij}v_{i+1,j}; 1 \leq j \leq m),
\]
Where \(v_{ij}v_{i,j+1}\) represent an edge between \(v_{ij}\) and \(v_{i,j+1}\) on the \(i^{th}\) layer; and \(v_{ij}v_{i+1,j}\) represent an edge joining the \(j^{th}\) vertices of the layers \(i\) and \(i + 1\).

The prism graph \(Y^n_m\) can be colored totally with five colors, say \(\{1,2,3,4,5\}\) as follows:

In coloring we use the notations \(S_1, S_2, S_3, S_4\) and \(S_5\) for the set of colors \(\{1,2,3,4,5\}, \{2,3,4,5,1\}, \{3,4,5,1,2\}, \{4,5,1,2,3\}\) and \(\{5,1,2,3,4\}\) respectively.

There are five cases depending on the value of \(m\).

(i) \(m \equiv 0 (mod\ 5)\)
(ii) \(m \equiv 1 (mod\ 5)\)
(iii) \(m \equiv 2 (mod\ 5)\)
(iv) \(m \equiv 3 (mod\ 5)\)
(v) \(m \equiv 4 (mod\ 5)\)

Case (i): Let \(m \equiv 0 (mod\ 5)\). The value of \(n\) is arbitrary.
For \(1 \leq i \leq n, 1 \leq j \leq m\)
Define the function \(f_1\) from the vertices of \(Y^n_m\) to the color set \(\{1,2,3,4,5\}\) as follows:

\[
f_1(v_{ij}) = \begin{cases} 
S_\alpha^1, & \text{if } i \equiv 1 (mod\ 5) \\
S_\alpha^2, & \text{if } i \equiv 2 (mod\ 5) \\
S_\alpha^3, & \text{if } i \equiv 3 (mod\ 5) \\
S_\alpha^4, & \text{if } i \equiv 4 (mod\ 5) \\
S_\alpha^5, & \text{if } i \equiv 0 (mod\ 5) 
\end{cases}
\]
where the superscript \(\alpha\) defines the number of times the \(S_i\) to be repeated according to the value \(\alpha = \frac{m}{5}\).

Define the function \(f_2\) from the edges on the layers of the function to the color set \(\{1,2,3,4,5\}\) as follows:

\[
f_2(v_{ij}v_{i,j+1}) = \begin{cases} 
S_i, & \text{if } i \equiv 1 (mod\ 5) \\
S_i, & \text{if } i \equiv 2 (mod\ 5) \\
S_i, & \text{if } i \equiv 3 (mod\ 5) \\
S_i, & \text{if } i \equiv 4 (mod\ 5) \\
S_i, & \text{if } i \equiv 0 (mod\ 5) 
\end{cases}
\]

By using the above defined coloring pattern, the graph \(Y^n_m\) is total colored and the total-chromatic number of the prism graph of \(n\)-layers, \(\chi_t(Y^n_m)\) is 5.

Case (ii): Let \(m \equiv 1 (mod\ 5)\). The value of \(n\) is arbitrary.
For \(1 \leq i \leq n, 1 \leq j \leq m\)
Define the function \(f_1\) from the vertices of \(Y^n_m\) to the color set \(\{1,2,3,4,5\}\) as follows:

\[
f_1(v_{ij}) = \begin{cases} 
S_i^1, & \text{if } i \equiv 1 (mod\ 5) \\
S_i^2, & \text{if } i \equiv 2 (mod\ 5) \\
S_i^3, & \text{if } i \equiv 3 (mod\ 5) \\
S_i^4, & \text{if } i \equiv 4 (mod\ 5) \\
S_i^5, & \text{if } i \equiv 0 (mod\ 5) 
\end{cases}
\]

Define the function \(f_2\) from the edges on the layers of the function to the color set \(\{1,2,3,4,5\}\) as follows:

\[
f_2(v_{ij}v_{i,j+1}) = \begin{cases} 
S_i^1, & \text{if } i \equiv 1 (mod\ 5) \\
S_i^2, & \text{if } i \equiv 2 (mod\ 5) \\
S_i^3, & \text{if } i \equiv 3 (mod\ 5) \\
S_i^4, & \text{if } i \equiv 4 (mod\ 5) \\
S_i^5, & \text{if } i \equiv 0 (mod\ 5) 
\end{cases}
\]

By using the above defined coloring pattern, the graph \(Y^n_m\) is total colored and the total-chromatic number of the prism graph of \(n\)-layers, \(\chi_t(Y^n_m)\) is 5.

Case (iii): Let \(m \equiv 2 (mod\ 5)\). The value of \(n\) is arbitrary.
For \(1 \leq i \leq n, 1 \leq j \leq m\)
Define the function \(f_1\) from the vertices of \(Y^n_m\) to the color set \(\{1,2,3,4,5\}\) as follows:

\[
f_1(v_{ij}) = \begin{cases} 
S_i^1, & \text{if } i \equiv 1 (mod\ 5) \\
S_i^2, & \text{if } i \equiv 2 (mod\ 5) \\
S_i^3, & \text{if } i \equiv 3 (mod\ 5) \\
S_i^4, & \text{if } i \equiv 4 (mod\ 5) \\
S_i^5, & \text{if } i \equiv 0 (mod\ 5) 
\end{cases}
\]

Define the function \(f_2\) from the edges on the layers of the function to the color set \(\{1,2,3,4,5\}\) as follows:

\[
f_2(v_{ij}v_{i,j+1}) = \begin{cases} 
S_i^1, & \text{if } i \equiv 1 (mod\ 5) \\
S_i^2, & \text{if } i \equiv 2 (mod\ 5) \\
S_i^3, & \text{if } i \equiv 3 (mod\ 5) \\
S_i^4, & \text{if } i \equiv 4 (mod\ 5) \\
S_i^5, & \text{if } i \equiv 0 (mod\ 5) 
\end{cases}
\]

By using the above defined coloring pattern, the graph \(Y^n_m\) is total colored and the total-chromatic number of the prism graph of \(n\)-layers, \(\chi_t(Y^n_m)\) is 5.

Case (iv): Let \(m \equiv 3 (mod\ 5)\). The value of \(n\) is arbitrary.
For \(1 \leq i \leq n, 1 \leq j \leq m\)
Define the function \(f_1\) from the vertices of \(Y^n_m\) to the color set \(\{1,2,3,4,5\}\) as follows:

\[
f_1(v_{ij}) = \begin{cases} 
S_i^1, & \text{if } i \equiv 1 (mod\ 5) \\
S_i^2, & \text{if } i \equiv 2 (mod\ 5) \\
S_i^3, & \text{if } i \equiv 3 (mod\ 5) \\
S_i^4, & \text{if } i \equiv 4 (mod\ 5) \\
S_i^5, & \text{if } i \equiv 0 (mod\ 5) 
\end{cases}
\]

Define the function \(f_2\) from the edges on the layers of the function to the color set \(\{1,2,3,4,5\}\) as follows:

\[
f_2(v_{ij}v_{i,j+1}) = \begin{cases} 
S_i^1, & \text{if } i \equiv 1 (mod\ 5) \\
S_i^2, & \text{if } i \equiv 2 (mod\ 5) \\
S_i^3, & \text{if } i \equiv 3 (mod\ 5) \\
S_i^4, & \text{if } i \equiv 4 (mod\ 5) \\
S_i^5, & \text{if } i \equiv 0 (mod\ 5) 
\end{cases}
\]
where the superscript \( \alpha \) defines the number of times the \( S_z \) to be repeated according to the value \( \alpha = \frac{m-3}{5} \).

Define the function \( f_2 \) from the edges on the layers of the function to the color set \{1,2,3,4,5\} as follows:

\[
f_2(v_{ij}, v_{i,j+1}) = \begin{cases} 
S_4^1312, & i \equiv 1 \pmod{5} \\
S_4^2135, & i \equiv 2 \pmod{5} \\
S_4^3514, & i \equiv 3 \pmod{5} \\
S_4^4245, & i \equiv 4 \pmod{5} \\
S_4^5432, & i \equiv 0 \pmod{5} 
\end{cases}
\]

Define the function \( f_3 \) from the edges joining the vertices of adjacent layers to the color set \{1,2,3,4,5\} as follows:

\[
f_3(v_{ij}, v_{i+1,j}) = \begin{cases} 
S_5^1444, & i \equiv 1 \pmod{5} \\
S_5^2222, & i \equiv 2 \pmod{5} \\
S_5^3131, & i \equiv 3 \pmod{5} \\
S_5^4524, & i \equiv 4 \pmod{5} \\
S_5^5555, & i \equiv 0 \pmod{5} 
\end{cases}
\]

By using the above defined coloring pattern, the graph \( Y_m^m \) is total colored and the total-chromatic number of the prism graph of \( n \)-layers, \( \chi_t(Y_m^m) \) is 5.

**Case (v):** Let \( m \equiv 4 \pmod{5} \). The value of \( n \) is arbitrary. For \( 1 \leq i \leq n, 1 \leq j \leq m \)

Define the function \( f_1 \) from the vertices of \( Y_m^m \) to the color set \{1,2,3,4,5\} as follows:

\[
f_1(v_{ij}) = \begin{cases} 
S_4^11234, & i \equiv 1 \pmod{5} \\
S_4^23452, & i \equiv 2 \pmod{5} \\
S_4^35131, & i \equiv 3 \pmod{5} \\
S_4^42423, & i \equiv 4 \pmod{5} \\
S_4^54515, & i \equiv 0 \pmod{5} 
\end{cases}
\]

where the superscript \( \alpha \) defines the number of times the \( S_z \) to be repeated according to the value \( \alpha = \frac{m-4}{5} \).

Define the function \( f_2 \) from the edges on the layers of the function to the color set \{1,2,3,4,5\} as follows:

\[
f_2(v_{ij}, v_{ij+1}) = \begin{cases} 
S_4^13152, & i \equiv 1 \pmod{5} \\
S_4^21235, & i \equiv 2 \pmod{5} \\
S_4^34523, & i \equiv 3 \pmod{5} \\
S_4^45314, & i \equiv 4 \pmod{5} \\
S_4^52341, & i \equiv 0 \pmod{5} 
\end{cases}
\]

Define the function \( f_3 \) from the edges joining the vertices of adjacent layers to the color set \{1,2,3,4,5\} as follows:

\[
f_3(v_{ij}, v_{i+1,j}) = \begin{cases} 
S_5^14541, & i \equiv 1 \pmod{5} \\
S_5^22314, & i \equiv 2 \pmod{5} \\
S_5^31245, & i \equiv 3 \pmod{5} \\
S_5^43152, & i \equiv 4 \pmod{5} \\
S_5^55423, & i \equiv 0 \pmod{5} 
\end{cases}
\]

By using the above defined coloring pattern, the graph \( Y_m^m \) is total colored and the total-chromatic number of the prism graph of \( n \)-layers, \( \chi_t(Y_m^m) \) is 5.

**Illustration 3:** Consider a prism graph of 5-layers, \( Y_5^5 \).
This illustration comes under case 1.

Hence by using the general color pattern of the above theorem, we assign the colors to the vertices and the edges as shown in the fig. 3. The total chromatic number is \( \chi_t(G) = 5 \).

**Illustration 4:** Consider a prism graph of 6-layers, \( Y_6^6 \).
This illustration comes under case 2.

Hence by using the general color pattern of the above theorem, we assign the colors to the vertices and the edges as shown in the fig 4. The total chromatic number is \( \chi_t(G) = 5 \).

**Illustration 5:** Consider a prism graph of 5-layers, \( Y_5^5 \).
This illustration comes under case 3.

Hence by using the general color pattern of the above theorem, we assign the colors to the vertices and the edges as shown in the fig 5. The total chromatic number is \( \chi_t(G) = 5 \).
Illustration 6: Consider a prism graph of 4-layers, \(Y_3^4\).
This illustration comes under case 4.

Illustration 7: Consider a prism graph of 5-layers, \(Y_3^5\).
This illustration comes under case 5.

Hence by using the general color pattern of the above theorem, we assign the colors to the vertices and the edges as shown in the fig. 6. The total chromatic number is \(\chi_t(G) = 5\).

III. TOTAL COLORING OF GRID GRAPHS

Theorem 2: The grid graph \(P_m \times P_n\) \((m, n \geq 3)\) admits total coloring and its total chromatic number is 5.

Proof: If the vertex sets of the paths \(P_m\) and \(P_n\) are \(\{u_1, u_2, \ldots, u_m\}\) and \(\{w_1, w_2, \ldots, w_n\}\) respectively, then the vertex set of the grid graph \(P_m \times P_n\) is given by

\[
V(P_m \times P_n) = \bigcup_{i=1}^{m}(v_{ij}; 1 \leq j \leq n)
\]

and its edge set is given by

\[
E(P_m \times P_n) = \bigcup_{i=1}^{m}(v_{ij}v_{i+1,j}, 1 \leq j \leq n) \bigcup_{j=1}^{m}(v_{ij}v_{i,j+1}, 1 \leq i \leq m)
\]

For \(1 \leq i \leq m, 1 \leq j \leq n\)

Define the function \(f_i\) from the vertices of the grid graph \(P_m \times P_n\) to the color set \(\{1, 2, 3, 4, 5\}\) as follows:

\[
f_i(v_{ij}) \equiv \begin{cases} 
(i + j) \, \text{mod} \, 5, & \text{if } (i + j) \neq 0 \, \text{mod} \, 5 \\
5, & \text{if } (i + j) \equiv 0 \, \text{mod} \, 5
\end{cases}
\]

Define the function \(f_2\) from the edges of the grid graph \(P_m \times P_n\) to the color set \(\{1, 2, 3, 4, 5\}\) as follows:

\[
f_2(v_{ij}v_{i,j+1}) \equiv \begin{cases} 
(i + j - 1) \, \text{mod} \, 5, & \text{if } (i + j - 1) \neq 0 \, \text{mod} \, 5 \\
5, & \text{if } (i + j - 1) \equiv 0 \, \text{mod} \, 5
\end{cases}
\]

By using the above pattern of coloring, the grid graph can be given total colored and the total-chromatic number of the grid graph, \(\chi_t(P_m \times P_n)\) is 5, for \(m, n \geq 3\).

Illustration 8: Consider the grid graph \(P_3 \times P_5\).

Hence by using the coloring pattern given in the above theorem, we assign the colors to the vertices and the edges as shown in the fig. 8. The total chromatic number is \(\chi_t(G) = 5\).

IV. CONCLUSION

The total coloring of a prism graph of \(n\)-layers and the grid graph were discussed and found that the total coloring chromatic number of both them to be 5.

REFERENCES

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