Special Pairs of Pythagorean Triangle

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Abstract— we illustrate the different methods of obtaining pairs of Pythagorean triangles which are such that, in each pair, the sum of the product of their generators is a perfect square. Also a few interesting properties among the pairs of Pythagorean triangles and special polygonal numbers are exhibited.

Keywords: Pair of Pythagorean triangles, special polygonal numbers

I. INTRODUCTION

The fascinating branch of mathematics is the theory of numbers where in Pythagorean triangles have been a matter of interest to various mathematicians and to the lovers of mathematics, because it is a treasure house in which the search for many hidden connections is a treasure hunt. For a rich variety of fascinating problems one may refer [1-17]. A careful observer of patterns may note that there is one to one correspondence between the polygonal numbers and the number of sides of the polygon.

In [18-20], special Pythagorean triangles connected with polygonal numbers and Nasty numbers are obtained. In [21] pairs of distinct Pythagorean triangles such that in each pair the difference between their perimeter is represented by

\[ kx^2, n > 2 \]

\[ \alpha \]

In this communication, we present different methods of obtaining pairs of Pythagorean triangles which are such that, in each pair, the sum of the product of their generators is a perfect square. A few interesting properties among the pairs of Pythagorean triangles and special polygonal numbers are exhibited.

II. NOTATION USED

- \( t_{m,n} \): Polygonal number of rank \( n \) with size \( m \).
- \( P_{n}^{m} \): Pyramidal number of rank \( n \) with size \( m \).
- \( gn_{a} \): Gnomonic number of rank \( a \).
- \( J_{n} \): Jacobsthal number of rank \( n \).
- \( J_{n} \): Jacobsthal-Lucas number of rank \( n \).
- \( Pt_{n} \): Pantatope number of rank \( n \).
- \( Ky_{n} \): Kyneya number of rank \( n \).
- \( Pr_{n} \): Pronic number of rank \( n \).

III. METHOD OF ANALYSIS

Let \( T_{1}(x_{1}, y_{1}, z_{1}) \) and \( T_{2}(x_{2}, y_{2}, z_{2}) \) be two distinct Pythagorean triangles, where

\[ x_{1} = m^2 - n^2, \quad y_{1} = 2mn, \quad z_{1} = m^2 + n^2 \quad (1) \]

\[ x_{2} = p^2 - q^2, \quad y_{2} = 2pq, \quad z_{2} = p^2 + q^2 \]

\( m, n (m > n > 0) \) and \( p, q (p > q > 0) \) are the generators of \( T_{1} \) and \( T_{2} \) respectively.

We illustrate below the process of obtaining pairs of Pythagorean triangles such that the sum of the product of the generators is a perfect square.

\[ \text{ie, } mn + pq = \alpha^2 \]

(2)

Different choices of \( T_{1} \) and \( T_{2} \) are obtained by solving (2) through employing different forms of linear transformations for the generators \( m, n, p \) and \( q \) in (2).

CHOICE: I

Introducing the linear transformations

\[ m = u + v, \quad n = u - v, \quad p = r + u, \quad q = r - u, \quad u \neq v \neq r \]

in (2), it gives

\[ r^2 = \alpha^2 + v^2 \]

(3)

which is the well known Pythagorean equation which is satisfied by

\[ v = 2ab, \quad \alpha = a^2 - b^2, \quad r = a^2 + b^2, \quad a > b > 0 \]

Substituting the values of \( v \) and \( r \) in (3) and using (1), the corresponding sides of \( T_{1} \) and \( T_{2} \) are as follows

\[ x_{1}(a, b, u) = 8ab \]

\[ y_{1}(a, b, u) = 2(u^2 - 4a^2b^2) \]

\[ z_{1}(a, b, u) = 2(u^2 + 4a^2b^2) \]

\[ x_{2}(a, b, u) = 4(a^2 + b^2) \]

\[ y_{2}(a, b, u) = 2[(a^2 + b^2)^2 - u^2] \]

\[ z_{2}(a, b, u) = 2[(a^2 + b^2)^2 + u^2] \]

Properties:

- \( x_{1}(a, b, u(u + 1)) + x_{2}(a, b, u(u + 1)) = 8t_{3, u} * t_{4, a, a + b} \)
- \( 2Pr_{a} x_{2}(a, a + 1, u) - gn_{a} x_{1}(a, a + 1, u) = 0(\text{mod } 2a^2) \)
- \( 3\{y_{1}(a, b, u) + z_{1}(a, b, u) + z_{2}(a, b, u) - y_{2}(a, b, u)\} \)

is a Nasty number.

- \( x_{1}(a, a, (a + 1)) - y_{1}(a, a, (a + 1)) + z_{1}(a, a, (a + 1)) - 16Pr_{a}^5 \)

is a biquadratic integer.

- \( x_{1}(a, (a + 1)(a + 2), a + 3) + y_{1}(a, (a + 1)(a + 2), a + 3) + z_{1}(a, (a + 1)(a + 2), a + 3) = 192Pr_{a} + 4t_{4, a + 3} \)

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Note: 1

The substitution of (4) may also be taken as
\[ v = a^2 - b^2, \alpha = 2ab, r = a^2 + b^2, a > b > 0 \]

For this choice, the corresponding sides of \( T_1 \) and \( T_2 \) are respectively as follows

\[
\begin{align*}
\alpha^2 - r^2 &= 4v^2 + 2vs \\
y_1(v, s) &= 8v + 2s \\
z_1(v, s) &= 2(4v^2 + 2vs + 2vs) \\
x_2(v, s) &= -8v^2 + 8v \\
y_2(v, s) &= 2(v^2 - 2vs - 3s^2) \\
z_2(v, s) &= 2(5s^2 + v^2 - 2vs)
\end{align*}
\]

Properties:
- \( x_2(a, b, b + 1) - x_1(a, b, b + 1) = 16P_0^4 \)
- \( y_1(2^a + 1, I, u) + y_2(2^a + 1, u, l) = 8(Ky_a + 2) \)
- \( 3(2_{2n})^2 x_2(2^n, I, l) = (2_{j2n})^2 x_2(2^n, I, l) \)
- \( y_1(2^a + 1, u, u + 1) + z_1(2^a + 1, u, u + 1) + y_2(2^a + 1, u, u + 1) + z_2(2^a + 1, u + 1) + 4P_{3a}^2 + 4(2_{j2n})^2 \)

**CHOICE: 2**

Introduction of the linear transformations
\[
m = 2v + 2s, n = 2s, p = r + 2s, q = r - 2s, r > 2s > 0an in (2) leads to
\[
\alpha^2 - r^2 = 4v^2
\]

which is satisfied by \( \alpha = v + s, r = v - s \)

Substituting the value in (5) and using (1), the corresponding sides of \( T_1 \) and \( T_2 \) are as follows

\[
\begin{align*}
x_1(v, s) &= 4v + 2s \\
y_1(v, s) &= 8v + 2s \\
z_1(v, s) &= 4v^2 + 2vs + 2vs \\
x_2(v, s) &= -8v^2 + 8v \\
y_2(v, s) &= 2(v^2 - 2vs - 3s^2) \\
z_2(v, s) &= 2(5s^2 + v^2 - 2vs)
\end{align*}
\]

Properties:
- \( x_1(13v, 13v + 2) + y_1(13v, 13v + 2) = 384Pt_v + 16t_{3v} + 8P_{2a}^2 \)
- \( y_1(2v - 1, v) + x_2(2v - 1, v) = 16SO_v \)
- \( x_2(v, s(s + 1)(s + 2) + y_2(v, s(s + 1)(s + 2)) + z_2(v, s(s + 1)(s + 2) + 24j)^3 = 0\) \( (mod 4) \)
- \( 6|y_1(v, v) - x_1(v, v) \) is a Nasty number.

Note: 2

(6) is equivalent to the following systems of equations

\[
\begin{align*}
\alpha + r &= 2v \\
\alpha - r &= 2v \\
\alpha + r &= vs \\
\alpha - r &= 4
\end{align*}
\]

Now solving the system of equations (i), we have
\[ \alpha = vs + 1, r = vs - 1, \]
\[\alpha = 2^{2k}[x^2 + y^2], r = 2^{2k+1}xy, s = 2^k(x^2 - y^2)\]

Thus, in view of (6) and using (1), the corresponding sides of \(T_1\) and \(T_2\) are

\[x_1(x,y,k) = 2^{2k+2}(2^{4k} + 2^{2k+1})(x^2 - y^2)^2\]
\[y_1(x,y,k) = 2^{2k+2}(2^{2k} + 1)(x^2 - y^2)^2\]
\[z_1(x,y,k) = 2^{2k+2}(2^{4k} + 2^{2k+1} + 2)(x^2 - y^2)^2\]
\[x_2(x,y,k) = 2^{3k+3}xy(x^2 - y^2)\]
\[y_2(x,y,k) = 2^{4k+3}x^2y^2 - 2^{2k+1}(x^2 - y^2)^2\]
\[z_2(x,y,k) = 2^{24k+2}x^2y^2 + 2^{2k}(x^2 - y^2)^2\]

**CHOICE: 4**

Introducing the linear transformations

\[m = 4u + v, n = 4u - v, p = 3u + r, q = 3u - r, u \neq v, v \neq u (8)\]
in (2), it leads to

\[(5u)^2 = \alpha^2 + v^2 + r^2\]

which is satisfied by

\[\alpha = a^2 - b^2 - c^2, r = 2ab, v = 2ac, u = \frac{1}{5}(a^2 + b^2 + c^2)\]

Substituting as \(5A, b=5B\) and \(c=5C\) in the above equations, we get

\[\alpha = 25(A^2 - B^2 - C^2), r = 50AB,\]
\[v = 50AC, u = 5(A^2 + B^2 + C^2)\]

In view of (8) and using (1), the corresponding sides of \(T_1\) and \(T_2\) are as follows

\[x_1(A,B) = 400AC(A^2 + B^2 + C^2)\]
\[y_1(A,B) = 2[(20A^2 + 20B^2 + 20C^2)^2 - 2500A^2C^2]\]
\[z_1(A,B) = 2[(20A^2 + 20B^2 + 20C^2)^2 + 2500A^2C^2]\]
\[x_2(A,B) = 150AB(A^2 + B^2 + C^2)\]
\[y_2(A,B) = 2[(15A^2 + 15B^2 + 15C^2)^2 - 2500A^2B^2]\]
\[z_2(A,B) = 2[(15A^2 + 15B^2 + 15C^2)^2 + 2500A^2B^2]\]

**CHOICE: 5**

Note that (2) is satisfied by

\[m = a^2 - b^2, n = a^2 - b^2 - 4ab,\]
\[p = 2a^2 + 2ab, q = 2ab - 2b^2, a > 4b > 0\]

Hence the corresponding sides of \(T_1\) and \(T_2\) are obtained as

\[x_1(a,b) = 8ab(a^2 - b^2) - 16a^2b^2\]
\[y_1(a,b) = 2(a^2 - b^2)^2 - 4ab(a^2 - b^2)\]
\[z_1(a,b) = 2(a^2 + b^2)^2 + 8a^2b^2 - 8ab(a^2 - b^2)\]
\[x_2(a,b) = 4(a^4 - b^4 - 2a^3b + 2ab^3)\]
\[y_2(a,b) = 8ab(a^2 - b^2)\]
\[z_2(a,b) = 4(a^4 + b^4 + 2a^2b^2 + 2a^3b - 2ab^3)\]

**IV. CONCLUSION**

One may search for pairs of Pythagorean triangles such that the sum of the product of generators is represented by special polygonal numbers and pyramidal numbers.

**REFERENCES**