On the Non-Homogeneous Quintic Equation with Seven Unknowns

\[ xy(x^2 + y^2) + zw(z^2 + w^2) = (X^2 + Y^2)T^3 \]

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Abstract—We obtain infinitely many non-zero integer solutions \((x, y, z, w, X, Y, T)\) satisfying the non-homogeneous quintic equation with seven unknowns given by

\[ xy(x^2 + y^2) + zw(z^2 + w^2) = (X^2 + Y^2)T^3 \]. Various interesting relations between the solutions and special numbers, namely, polygonal numbers, Pyramidal numbers, Stella Octagonal numbers, Octahedral numbers, Jacobsthal number, Jacobsthal-Lucas number, keynea number, Centered pyramidal numbers are presented.

Index Terms—Centered pyramidal numbers, Integral solutions, Non-homogeneous Quintic equation, Polygonal numbers, Pyramidal numbers

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Notations:
- \( T_{m,n} \) - Polygonal number of rank \( n \) with size \( m \)
- \( SO_n \) - Stella Octagonal number of rank \( n \)
- \( J_n \) - Jacobsthal number of rank \( n \)
- \( KY_n \) - Keynea number of rank \( n \)
- \( CP_{n,6} \) - Centered hexagonal pyramidal number of rank \( n \)
- \( P_n^m \) - Pyramidal number of rank \( n \) with size \( m \)
- \( OH_n \) - Octahedral number of rank \( n \)
- \( j_n \) - Jacobsthal-Lucas number of rank \( n \)

I. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-3]. For illustration, one may refer [4-10] for homogeneous and non-homogeneous quintic equations with three, four and five unknowns. This paper concerns with the problem of determining non-trivial integral solution of the non-homogeneous quintic equation with seven unknowns given by

\[ xy(x^2 + y^2) + zw(z^2 + w^2) = (X^2 + Y^2)T^3 \]. A few relations between the solutions and the special numbers are presented.

Initially, the following two sets of solutions in \((x, y, z, w, X, Y, T)\) satisfy the given equation:

\[
(2k^2 + k + p, 2k^2 + k - p, p - 2k^2 + k, \\
p + 2k^2 - k, 2k, 4k^2, 2k) \\
(2k^2 + 3k + p + 1, 2k^2 + 3k - p + 1, p - 2k^2 - k, \\
p + 2k^2 + k, 2k + 1, 4k^2 + 4k + 1, 2k + 1)
\]

However, we have other patterns of solutions, which are illustrated below:

II. METHOD OF ANALYSIS

The Diophantine equation representing the non-homogeneous quintic equation is given by

\[ xy(x^2 + y^2) + zw(z^2 + w^2) = (X^2 + Y^2)T^3 \] (1)

Introduction of the transformations

\[ x = u + p, y = u - p, z = p + v, w = p - v, \]

\[ X = u + v, Y = u - v \] (2)

in (1) leads to

\[ u^2 - v^2 = T^3 \] (3)

The above equation (3) is solved through different approaches and thus, one obtains different sets of solutions to (1).

A. Approach I

The solution to (3) is obtained as

\[
\begin{align*}
u & = a(a^2 - b^2) \\
v & = b(a^2 - b^2) \\
T & = (a^2 - b^2)
\end{align*}
\] (4)

In view of (2) and (4), the corresponding values of \(x, y, z, w, X, Y, T\) are represented by

\[
\begin{align*}
x & = a(a^2 - b^2) + p \\
y & = a(a^2 - b^2) - p \\
z & = p + b(a^2 - b^2) \\
w & = p - b(a^2 - b^2) \\
X & = (a^2 - b^2)(a + b) \\
Y & = (a^2 - b^2)(a - b) \\
T & = a^2 - b^2
\end{align*}
\] (5)
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\[ xy(x^2 + y^2) + zw(z^2 + w^2) = (X^2 + Y^2)T^3 \]

The above values of \( x, y, z, w, X, Y \) and \( T \) satisfies the following relations:

1. \( [x(a,b) + y(a,b)][X(a,b) + Y(a,b)] \) is a perfect square.

2. The following expressions are nasty numbers:
   
   \[(a) \; 3[X(a(a + 1), a) + Y(a(a + 1), a) - 4T_{3,a} - 6CP_{a,6}]
   
   \[(b) \; 3[x(a(a + 1), a) + y(a(a + 1), a) + z(a(a + 1), a) - w(a(a + 1), a)]
   
   \[(c) \; 2(x(2a, a) + y(2a, a) - 6SO_a + 12T_{3,a}].
   
3. The following expressions are cubic integers:
   
   \[(a) \; 9[x(2a, a) + y(2a, a) + z(2a, a) + w(2a, a) + X(2a, a) + Y(2a, a)]
   
   \[(b) \; 4[Y(a, 1) - X(a, 1) + 2T_{4,a}]
   
4. \( w(a, 1) - z(a, 1) + T(\alpha, 1) + T_{4,a} = 1 \)

5. \( x(2a, a) + y(2a, a) + T(\alpha, 2a) = 12CP_{a, 6} + 6T_{3,a} - 3SO_a \)

6. \( T(2a, a) + z(2a, a) + w(2a, a) - 3T_{3,a} = 0 \mod 2 \)

7. \( T(2^n + 1, 2^n) + 3J_{2n+1} = 3KY_{2n} \)

B. Approach2

The assumptions

\[ u = UT, \; v = VT \]

in (3) yields to

\[ U^2 - V^2 = T \]

(i) Taking \( T = -t^2 \)

in (7), we get

\[ U^2 + t^2 = V^2 \]

Then the solution to (9) is given by

\[ t = 2\alpha\beta, V = \alpha^2 + \beta^2, U = \alpha^2 - \beta^2, \alpha > \beta > 0 \]

\[ U = 2\alpha\beta, V = \alpha^2 + \beta^2, t = \alpha^2 - \beta^2, \alpha > \beta > 0 \]

From (6), (8) and (10) we get,

\[
\begin{align*}
6.1 \quad (8a^2 - \beta^2)(8a^2 + \beta^2) \\
v &= -4a^2\beta^2(\alpha^2 + \beta^2) \\
T &= -4a^2\beta^2
\end{align*}
\]

In view of (12) and (2), we get the corresponding integral solution of (1) as

\[
\begin{align*}
x &= -4a^2\beta^2(\alpha^2 - \beta^2) + p \\
y &= -4a^2\beta^2(\alpha^2 - \beta^2) - p \\
z &= p - 4a^2\beta^2(\alpha^2 + \beta^2) \\
w &= p + 4a^2\beta^2(\alpha^2 + \beta^2) \\
X &= -8a^4\beta^2 \\
Y &= 8a^2\beta^4 \\
T &= -4a^2\beta^2
\end{align*}
\]
\[ U = \cos \frac{n\pi}{2} (a^2 - b^2) - 2ab \sin \frac{n\pi}{2} \]
\[ t = 2ab \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2} (a^2 - b^2) \]

In view of (2), (6), (8) and (18), the corresponding values of \( x, y, z, w, X, Y, T \) are represented as

\[ x = T[\cos \frac{n\pi}{2} (a^2 - b^2) - 2ab \sin \frac{n\pi}{2}] + p \]
\[ y = T[\cos \frac{n\pi}{2} (a^2 - b^2) - 2ab \sin \frac{n\pi}{2}] - p \]
\[ z = p + a^2 + b^2 \]
\[ w = p - a^2 - b^2 \]
\[ X = T[\cos \frac{n\pi}{2} (a^2 - b^2) - 2ab \sin \frac{n\pi}{2} + a^2 + b^2] \]
\[ Y = T[\cos \frac{n\pi}{2} (a^2 - b^2) - 2ab \sin \frac{n\pi}{2} - a^2 - b^2] \]
\[ T = -(\sin \frac{n\pi}{2} (a^2 - b^2) + 2ab \cos \frac{n\pi}{2}) \]

(iii) 1 can also be written as
\[ 1 = \frac{(m^2 - n^2) + 2mn((m^2 - n^2) - i2mn)}{(m^2 + n^2)^2} \]  
(20)

Substituting (16) and (20) in (14) and using the method of factorization, define,
\[ (U + iT) = \frac{(m^2 - n^2) + i2mn(m^2 - n^2) - 4mnab)}{(m^2 + n^2)^2} (a + ib)^2 \]  
(21)

Equating real and imaginary parts in (21) we get
\[ U = \frac{1}{m^2 + n^2} \{(m^2 - n^2)(a^2 - b^2) - 4mnab\} \]
\[ t = \frac{1}{m^2 + n^2} \{2ab(m^2 - n^2) + 2mn(a^2 - b^2)\} \]
(22)

In view of (2), (6), (8) and (22), the corresponding values of \( x, y, z, w, X, Y, T \) are represented as
\[ x = (m^2 + n^2)T[(A^2 - B^2)(m^2 - n^2) - 4mnAB] + p \]
\[ y = (m^2 + n^2)T[(A^2 - B^2)(m^2 - n^2) - 4mnAB] - p \]
\[ z = p + (m^2 + n^2)^2(A^2 + B^2)T \]
\[ w = p - (m^2 + n^2)^2(A^2 - B^2)T \]
\[ X = (m^2 + n^2)T[(m^2 - n^2)(A^2 - B^2) - 4mnAB + (m^2 + n^2)(A^2 + B^2)] \]
\[ Y = (m^2 + n^2)T[(m^2 - n^2)(A^2 - B^2) - 4mnAB - (m^2 + n^2)(A^2 + B^2)] \]
\[ T = -(m^2 + n^2)^2[2AB(m^2 - n^2) + 2mn(A^2 - B^2)] \]
(23)

\[ (u - v)(u + v) = 1 \times T^3 \]  
(18)

Writing (24) as a set of double equations in two different ways as shown below, we get

**Set1:** \( u + v = T^3 \), \( u - v = 1 \)

**Set2:** \( u - v = T^3 \), \( u + v = 1 \)

Solving set1, the corresponding values of \( u, v \) and \( T \) are given by
\[ u = 4k^3 + 6k^2 + 3k + 1, v = 4k^3 + 6k^2 + 3k, \]
\[ T = 2k + 1 \]  
(25)

In view of (25) and (2), the corresponding solutions to (1) obtained from set1 are represented as shown below:
\[ x = 4k^3 + 6k^2 + 3k + 1 + p \]
\[ y = 4k^3 + 6k^2 + 3k + 1 - p \]
\[ z = p + 4k^3 + 6k^2 + 3k \]
\[ w = p - 4k^3 - 6k^2 - 3k \]
\[ X = 8k^3 + 12k^2 + 6k + 1 \]
\[ Y = 1 \]
\[ T = 2k + 1 \]

**Properties:**
1. \( x(a) + y(a) - 24P_a^4 = 0 \) (mod 2)
2. \( 2[x(a) + y(a) + \lambda(a) + w(a) + T(a) - 24P_a^4 - 6I_{4a} + 2T_{8a} - 2\alpha] \) is a nasty number.
3. \( 4[X(a) + Y(a) - 8P_a^8 - 16T_{3a} - 6(OH)_a + 4CP_{a,6}] \) is a cubic integer.
4. \( X(a) - Y(a) + x(a) + y(a) + \lambda(a) + w(a) - 24P_a^4 - 4CP_{a,6} + 2SO_a \equiv 0 \) (mod 4)
5. \( 4[x(a) + y(a) + Y(a) + T(a) - 48P_a^3 + 24T_{3a} - 12(OH)_a + 8CP_{a,6}] \) is a biquadratic integer

**Remark2:**
Similarly, the solutions corresponding to set2 can also be obtained.

**D. Approach4**

Substituting \( T = a^2 - b^2 \) in (3) and writing it as a system of double equations as
\[ u + v = (a + b)^3 \]
\[ u - v = (a - b)^3 \]
and solving, we get
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\[ xy(x^2 + y^2) + zw(z^2 + w^2) = (X^2 + Y^2)T^3 \]

\[
\begin{align*}
u &= a^3 + 3ab^2 \\
v &= 3a^2b + b^3
\end{align*}
\]

(26)

In view of (26) and (2), the corresponding solutions to (1) are represented as shown below:

\[
\begin{align*}
x &= a^3 + 3ab^2 + p \\
y &= a^3 + 3ab^2 - p \\
z &= p + 3a^2b + b^3 \\
w &= p - 3a^2b - b^3 \\
X &= a^3 + 3ab^2 + 3a^2b + b^3 \\
Y &= a^3 + 3ab^2 - 3a^2b - b^3 \\
T &= a^2 - b^2
\end{align*}
\]

Properties:

1. \( 4[x(a,a) + y(a,a) + z(a,a) - w(a,a)] \) is a cubic integer
2. \( X(a,1) + Y(a,1) - 4CP_{a,3} - 14T_{4,a} + 4T_{9,a} = 0 \)
3. \( x(a,a) + y(a,a) + z(a,a) - w(a,a) + X(a,a) + Y(a,a) - 12SO_{a} - 36T_{4,a} + 24T_{5,a} = 0 \)
4. \( 16j_6n - T(2^{2n},2^{2n}) - X(2^{2n},2^{2n}) - Y(2^{2n},2^{2n}) - x(2^{2n},2^{2n}) - y(2^{2n},2^{2n}) \)
   \( \text{is a biquadratic integer} \)
5. \( X(2a,a) + Y(2a,a) - 2x(2a,a) + 2y(2a,a) - 2T(2a,a) \equiv 0 \text{ (mod 20)} \)

III. CONCLUSION

In conclusion, one may search for different patterns of solutions to (1) and their corresponding properties.

REFERENCES

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