Accelerated Multipoint Root Finding Iterative Methods

Eglantina Kalluci, Fatmir Hoxha

Abstract*— Root finding is one of the most significant problems not only of applied mathematics, but also of engineering sciences, physics, finance etc. The implementation of efficient numerical methods to build-in functions in different software programs is a task we want to achieve. We possess different groups of methods with sufficiently good convergence order, but as we know the higher the speed is a larger amount of function and derivative evaluations per iteration is needed. In this paper we will present new multipoint methods with higher computational efficiency, than known ones. The comparison will be made by defining the computational efficiency based on the convergence order, and the efficiency index, which measures the cost of performing iteration.

***Index Terms*— efficiency index, iterative method, order of convergence, root finding.**

I. INTRODUCTION

The solution of transcendent equation is one of the most investigated topics in mathematics, and because of the missing general exact solvers, the numerical methods lead the top solvers for this class of equations. Related with this aim we can use a vast literature as for example Ostrowski (1960), Traub (1964), Ortega and Rheinboldt (1970), Neta , Osada (1994) etc. These methods can be divided into one-point and multipoint schemes. A lot of one-point iterative methods, members of Traub-Schroder sequence, which depends on f and its first *r-1* derivatives, cannot obtain an order higher than r . One of the indicators of the measuring the efficiency is the informational efficiency of one-point methods, expressed as the ratio of the order of convergence and the number of required function evaluations per iteration, cannot exceed 1. Multipoint methods give a great improvement concerning the convergence order and information and computation efficiency. During the last ten years have been published at least 200 multipoint methods (McNamee, 2007). Many of them turned out to be either inefficient or slight modifications/variations of already known methods. In a lot of other cases 'new' methods were only rediscovered methods. For this reason in this paper we are focused in making a systematic review of multipoint methods, concerning mainly on the most efficient methods.

II. THE CLASSIFICATION OF ITERATIVE ROOT FINDING METHODSP

Let f be a real function of a real variable. If $f(\alpha)=0$ then α is a root of the equation $f(x)=0$. The following classification of iterative methods is made based on the definitions presented by Traub (1964). We will assume that f has a certain number of continuous derivatives in the neighbourhood of the root α .

(a) Let a iterative method be of the form

$$x_{i+1}=\phi(x_i)\ \ (k=0,1,2,\cdots), \qquad (1)$$

where x_i is an approximation of the root α and ϕ is an iterative function. The iterative method starts with an initial approximation x_0 and at every step we use only the last known approximate. For this reason this is called one-point method.

(b) Next let x_{i+1} be determined by new information at x_i and reused information at $x_{i-1},\cdots,x_{i-n}$. Thus

$$x_{i+1}=\phi(x_i;x_{i-1},\cdots,x_{i-n}). \qquad (2)$$

Then ϕ is called one-point iterative function with memory. The semicolon in (2) separates the points at which new data are used from the points at which old data are reused. One example of iterative function with memory is the well known secant method

$$x_{i+1}=x_i-\frac{x_i-x_{i-1}}{f(x_i)-f(x_{i-1})}f(x_i),\ (i=1,2,\cdots). \qquad (3)$$

(c) Another type of iteration functions is defined by using the expression $\omega_1(x_i),\omega_2(x_i),\cdots,\omega_n(x_i)$, where x_i is the common argument. The iterative function ϕ , defined as

$$x_{i+1}=\phi(x_i,\omega_1(x_i),\omega_2(x_i),\cdots,\omega_n(x_i)), \qquad (4)$$

is called a multipoint iterative function without memory. The simplest examples are Steffensen's method

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$$x_{i+1} = x_i - \frac{f^2(x_i)}{f(x_i + f(x_i)) - f(x_i)} \tag{5}$$

with $\omega_1(x_i) = x_i + f(x_i)$,

and Traub-Steffensen's method

$$x_{i+1} = x_i - \frac{\gamma f^2(x_i)}{f(x_i + \gamma f(x_i)) - f(x_i)}$$

with $\omega_1(x_i) = x_i + \gamma f(x_i)$. (6)

(d)Finally, if x_{i+1} is determined by new information at $x_{i-1}, \cdots, x_{i-k}$ and reused information $x_{i-k-1}, \cdots, x_{i-n}$ the iterative function

$$x_{i+1} = \phi(x_i, x_{i-1}, \cdots, x_{i-k}; x_{i-k-1}, \cdots, x_{i-n}),\ n > k, \tag{7}$$

is called a multipoint iterative function with memory. The semicolon in (7) separates the points at which new data are used from the points at which old data are reused. There are no well-known methods of multipoint iterative functions with memory. In this paper we will treat some new improvements of multipoint methods without and with memory for finding a simple zero.

III. MULTIPOINT ITERATIVE METHODS

One of the most important features to determine the advantages of on iterative method is the convergence rate determined by the order of convergence. Let $\{x_i\}$ be a sequence that converges to α and let $\varepsilon_i = x_i - \alpha$. If there exists a real number *p* and a nonzero positive constant C_p such that

$$\lim_{i\to\infty} \frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} = C_p,$$

then *p* is called the order of the sequence $\{x_i\}$ and C_p is the asymptotic error constant. In practise, when testing new methods is of interest to use the computational order of convergence defined by

$$\tilde{r} = \frac{\log|(x_i - \alpha)/(x_{i-1} - \alpha)|}{\log|(x_{i-1} - \alpha)/(x_{i-2} - \alpha)|}, \tag{8}$$

where x_{i-2}, x_{i-1} and x_i are the last three successive approximations to the root α. Also this formula is only of theoretical value, since the value of the zero α is unknown in practice. Using the factorization $f(x) = (x - \alpha)g(x)$ and (8), is derived the approximate formula for computational order of convergence

$$r_c = \frac{\log|f(x_i)/f(x_{i-1})|}{\log|f(x_{i-1})/f(x_{i-2})|}, \tag{9}$$

which is of much better practical importance.

There are other measures for comparing various iterative techniques. Traub (1964) introduces the informational efficiency and efficiency index, which can be expressed in terms of the order (r) of the method and the number of function-(and derivative) evaluations (θ_f). The informational efficiency of an iterative method (M) is defined as

$$I(M) = \frac{r}{\theta_f}. \tag{10}$$

The efficiency index (or computational efficiency) is given by

$$E(M) = r^{1/\theta_f}, \tag{11}$$

the definition that was introduces by Ostrowski in (Ostrowski, 1960). One major goal in designing new numerical methods is to obtain a method with the best possible computational efficiency. Each memory-free iteration consists of

- new function evaluation, and
- arithmetic operations used to combine the available data.

Minimizing the total number of arithmetic operations through an iterative process which would provide the zero approximation of the desired accuracy, would be very much dependent on the particular properties of a function *f* whose zero is sought. Regarding the definition (10) and (11), this means that it is desirable to achieve as high as possible convergence order with the fixed number of function evaluations per iteration. Below we will give the Kung-Traub conjecture, which is used as a definition for an optimal multipoint iterative method.

Kung-Traub conjecture (1974). Multipoint iterative methods without memory, costing (n+1) function evaluations per iteration, have order of convergence at most 2^n.

Multipoint methods that satisfy Kung-Traub conjecture are called optimal methods, consequently, the optimal order is $r = 2^n$ so that the optimal efficiency index is

$$E_n^{(0)} = 2^{n/n+1}.$$

In the following we will consider the case of root-finding iterative methods without memory for simple roots that use Hermitian type of information (H-information). Most of optimal methods are based on H-information, an exception are Jarratt's families of two-point methods. Here we are recalling one old optimal method from the class of two-point methods, which is a generalization of Ostrowski method (1960) of order 3 and is King's family (1973) of order 4,

$$\begin{aligned}
(1)\, y_k &= x_k - \frac{f(x_k)}{f'(x_k)} \\
(2)\, x_k &= y_k - \frac{f(y_k)}{f'(x_k)} \frac{f(x_k) + \beta f(y_k)}{f(x_k) + (\beta - 2) f(y_k)}
\end{aligned} \tag{12}$$

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The other is a Jarrat-like method rediscovered by Basu (2008) of order 4.

$$\begin{aligned} &(1) y_k = x_k - \frac{2}{3}\frac{f(x_k)}{f'(x_k)} \\ &(2) x_{k+1} = x_k - \frac{1}{2}\frac{f(x_k)}{f'(x_k)} + \frac{f(x_k)}{f'(x_k) - 3f(y_k)} \end{aligned} \quad (13)$$

Once an optimal two-point method is stated, it is easy to construct optimal three-point method of order 8 that require 4-function evaluations per iteration.
Petkovic et al (2014) give the following three-point method

$$\begin{aligned} &(1) y_k = x_k - \frac{f(x_k)}{f'(x_k)} \\ &(2) z_k = \Phi_j(x_k, y_k) \\ &(3) x_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)} \end{aligned} \quad (14)$$

The first two steps has order $r_1 = 4$ and the third step has order $r_2 = 2$ and from Theorem 2.4 (Traub, 1964) on composite iterative methods yields that the order of convergence is $r = 8$, but this method is not optimal since it requires 5-function evaluation per iteration. The authors propose the use of Hermite's interpolating polynomial and the weight functions to reduce the function evaluation per iteration. In this paper we propose two new three-point methods. The first proposition is made using at the third step the third order method given from Homeier in (2005):

$$\begin{aligned} &(1) y_k = x_k - \frac{f(x_k)}{f'(x_k)} \\ &(2) z_k = \varphi_j(x_k, y_k) \\ &(3a) x_{k+1} = x_k - \frac{f(x_k)}{2}\left(\frac{1}{f'(x_k)} + \frac{1}{f'(y_k)}\right) \end{aligned} \quad (15)$$

The second proposition comes by using Ostrowski's the third order method (1960):

$$\begin{aligned} &(1) y_k = x_k - \frac{f(x_k)}{f'(x_k)} \\ &(2) z_k = \varphi_j(x_k, y_k) \\ &(3b) x_{k+1} = x_k - \frac{f(y_k)}{f'(x_k)}\frac{f(x_k)}{f(x_k) - 2f(y_k)} \end{aligned} \quad (16)$$

These methods are optimal, because they require 4 function evaluation per iteration and their order of convergence is

$$r = r_1 * r_2 = 12.$$

Other improvements of the methods (15) an (16) are made by using the approximation of the derivatives with divided differences and hence passing to multipoint methods with memory. Using these last techniques the order of convergence is increased and it follows another rule, different from the one used in this section.

Table I. The efficiency indexes of one, two and three-point methods.

Method	*n*	*r*	*I*	*E*
(12)	2	4	1.33	1.587
(13)	2	4	1.33	1.587
(14)	3	8	1.6	1.516
(14*)	3	8	2	1.682
(15)	3	12	3	1.861
(16)	3	12	3	1.861
(Newton method)	1	2	1	1

IV. CONCLUSIONS

In this paper we presented the advantages of multipoint iterative methods for solving nonlinear equations. We presented two well-known families of multipoint methods without memory and gave some new improvements in these families, which is reflected in the increasing values of the efficiency indexes of them. Extended research can be made in finding variations of multipoint methods with memory of the new ones.

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