Structure of Certain Ideals in Ternary Semirings

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Abstact: In this paper we studied about principal ternary ideals, simple ternary ideals and semisimple ternary ideals in ternary semirings.

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I. INTRODUCTION

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, and the like. The theory of ternary algebraic systems was introduced by D. H. Lehmer [9]. He investigated certain ternary algebraic systems called triplexes which turn out to be commutative ternary groups. D. Madhusudhana Rao[11] characterized the primary ideals in ternary semi groups, about T. K. Dutta and S. Kar [6] introduced and studied some properties of ternary semirings which is a generalization of ternary rings. D. Madhusudhana Rao and G. Srinivasarao [12] investigated and studied about special elements in a ternary semirings. Our main purpose in this paper is to introduce the Structure of ternary ideals in ternary semirings and in [13] they introduced the ternary semiring in which satisfies the some identities.

II. PRELIMINARIES

Definition* II.1**[13] **:** A nonempty set T together with a binary operation called addition and a ternary multiplication denoted by [ ] is said to be a ***ternary semiring if T is an additive commutative semigroup satisfying the following conditions :

i) $[[abc]de] = [a[bcd]e] = [ab[cde]]$,

ii) $[(a + b)cd] = [acd] + [bcd]$,

iii) $[a(b + c)d] = [abd] + [acd]$,

iv) $[ab(c + d)] = [abc] + [abd]$ for all $a; b; c; d; e \in$ T.

Throughout T will denote a ternary semiring unless otherwise stated.

***Note* II.2**: For the convenience we write $x_1x_2x_3$ instead of $\left[x_1x_2x_3\right]$

***Note* II.3**: Let T be a ternary semiring. If A,B and C are three subsets of T , we shall denote the set ABC = $\left\{\Sigma abc : a \in A, b \in B, c \in C\right\}$.

***Note* II.4**: Let T be a ternary semiring. If A,B are two subsets of T , we shall denote the set A + B = $\left\{a + b : a \in A, b \in B\right\}$.

***Note* II.5:** Any semiring can be reduced to a ternary semiring.

***Example* II.6:** Let T be an semigroup of all $m \times n$ matrices over the set of all non negative rational numbers. Then T is a ternary semiring with matrix multiplication as the ternary operation.

***Example* II.7:** Let $S = \{. . . ,-2i, -i, 0, i, 2i, . . . \}$ be a ternary semiring withrespect to addition and complex triple multiplication.

Example* II.8:** The set T consisting of a single element 0 with binary operation defined by 0 + 0 = 0 and ternary operation defined by 0.0.0 = 0 is a ternary semiring. This ternary semiring is called the ***null ternary semiring or the ***zero ternary semirings.***

***Example* II.9:** The set Q of all rational numbers with respect to ordinary addition and ternary multiplication [ ] defined by $[abc] = abc$ for all $a, b, c \in$ Q is a ternary semiring.

Definition* II.10** [13] : A ternary semiring T is said to be ***commutative ternary semiring provided $abc = bca = cab = bac = cba = acb$ for all $a,b,c \in$ T.

***Example* II.11:** $(Z^0, +, .)$ is a ternary semiring of infinite order which is commutative.

***Example* II.12:** The set 2I of all evern integers is a commutative ternary semiring with respect to ordinary addition and ternary multiplication [ ] defined by $[abc] = abc$ for all $a, b, c \in$ T.

Definition* II.13**[12] **:** A ternary semigroup (T , **.**) is said to be ***left regular , if it satisfies the identity $a = a^3xy \;\; \forall \; a , x , y \in T$.

Definition* II.14**[12] **:** A ternary semigroup (T , **.**) is said to be ***right regular , if it satisfies the identity $a = xya^3 \;\; \forall \; a , x , y \in T$

Definition* II.15** [12] **:** A ternary semigroup (T , **.**) is said to be ***lateral regular, if it satisfies the identify $a = xa^3y \;\; \forall \; a , x , y \in T$

Definition* II.16** [12] **:** A ternary semigroup (T , **.**) is said to be ***two sided regular, if it left as well as right regular.

Definition* II.17** [12] **:** A ternary semigroup (T , **.**) is said to be ***regular , *it is* left, lateral and right regular.

III. LEFT TERNARY IDEALS IN TERNARY SEMIRINGS

Definition* III.1**: A nonempty subset A of a ternary semiring T is said to be ***left ternary ideal or ***left ideal*** of T if

(1) $a, b \in$ A implies $a + b \in$ A.

(2) $b, c \in$ T, $a \in$ A implies $bca \in$ A.

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***Note* III.2**: A nonempty subset A of a ternary semigroup T is a left ideal of T if and only if A is additive subsemigroup of T and TTA $\subseteq$ A.

***Example* III.3:** In the ternary semiring Z^0 , nZ^0 is a left ideal for any $n \in$ N.

***Theorem* III.4: The nonempty intersection of any two left ideals of a ternary semiring T is a left ideal of T.**

***Theorem* III.5: The nonempty intersection of any family of left ideals of a ternary semiring T is a left ideal of T.**

***Theorem* III.6: The union of any two left ideals of a ternary semiring T is a left ideal of T.**

Theorem III.7: The union of any family of left ideals of a ternary semiring T is a left ideal of T.

We now introduce a maximal left ternary ideal and left ternary ideal generated by a subset of a ternary semiring.

Definition* III.8**: An ideal A of a ternary semiring T is said to be a ***maximal left ternary ideal or simply ***maximal left ideal*** provided A is a proper left ideal of T and is not properly contained in any proper left ideal of T.

Definition* III.9**: Let T be a ternary semiring and A be a non-empty subset of T. The smallest left ideal of T containing A is called ***left ternary ideal of T generated by A.

***Theorem* III.10: The left ideal of a ternary semiring T generated by a non-empty subset A is the intersection of all left ideals of T containing A.**

Proof: Let Δ be the set of all left ideals of T containing A. Since T itself is a left ideal of T containing A, T$\in \Delta$. So $\Delta \neq \varnothing$. Let $S^* = \bigcap_{S\in\Delta} S$ .Since A $\subseteq$S for all S $\in \Delta$, A $\subseteq$ S^*.

By theorem 3, 5, S^* is a left ideal of S. Let K be a left ideal of T containing A, K is a left ideal of T. Clearly A $\subseteq$ K. Therefore K $\in \Delta \Rightarrow S^* \subseteq$ K and hence $S^*$is the left ideal of T generated by A.

We now introduce a principal left ideal of a ternary semiring and characterize principal left ideal.

Definition* III.11** : A left ideal A of a ternary semiring T is said to be the ***principal left ternary ideal generated by a if A is a left ternary ideal generated by $\{a\}$ for some a $\in$ T. It is denoted by L (a) or $< a >_l$.

***Theorem* III.12: If T is a ternary semiring and *a* $\in$T then L(*a*) =**

$$\left\{\sum_{i=1}^{n} r_i t_i a + na : r_i, t_i \in T,\ n \in z_0^{\ +}\right\}.$$

Where Σ denotes a finite sum and $z_0^{\ +}$ is the set of all positive integer with zero.

Proof: Let A = $\left\{\sum_{i=1}^{n} r_i t_i a + na : r_i, t_i \in T,\ n \in z_0^{\ +}\right\}$. Let *a*, *b* ∈ A.

a, *b* ∈ A . Then *a* = $\sum r_i t_i a + na$ and *b* = $\sum r_j t_j a + na$ for r_i, t_i, r_j, t_j ∈ T, $n \in z_0^{\ +}$.

Now *a* + *b* = $\sum r_i t_i a + na$ + $\sum r_j t_j a + na$ $\Rightarrow$ *a* + *b* is a finite sum.

Therefore *a* + *b* ∈ A and hence A is a additive subsemigroup of T.

For t_1, t_2 ∈ T and *a* ∈ A.

Then $t_1 t_2 a = t_1 t_2 (\sum r_i t_i a + na)$

$= \sum r_i t_i (t_1 t_2 a) + n(t_1 t_2 a)$ ∈ A

Therefore $t_1 t_2 a \in$ A and hence A is a left ideal of T.

Let L be a left ideal of T containing *a*.

Let $r \in$ A. Then $r = \sum r_i t_i a + na$ for $r_i, t_i \in T,\ n \in z_0^{\ +}$.

If r = $\sum r_i t_i a + na$ ∈ L. Therefore A ⊆ L

and hence A is a smallest ideal containing *a*.

Therefore A = L (*a*) =

$$\left\{\sum_{i=1}^{n} r_i t_i a + na : r_i, t_i \in T,\ n \in z_0^{\ +}\right\}.$$

***Note* III.13**: if T is ternary semiring and $a \in$T then L(a) = $T^e T^e a + na$.

We now introduce a left simple ternary semigroup and characterize left simple ternary semigroups.

Definition* III.14**: A ternary semiring T is said to be ***left simple ternary semiring if T is its only left ternary ideal.

***Theorem* III.15: A ternary semiring T is a left simple ternary semiring if and only if TT*a* = T for all *a*$\in$T.**

Proof: Suppose that T is a left simple ternary semiring and *a* $\in$T. Let *s*, *v* $\in$TT*a*; *t*, *u* $\in$T

s, *v* $\in$ TT*a* $\Rightarrow s = \sum_{i=1}^{n} v_i w_i a$

$v = \sum_{j=1}^{n} v_j w_j a$ where v_i, v_j, w_i, w_j $\in$T, $n \in z_0^{\ +}$.

$s + v = \sum_{i=1}^{n} v_i w_i a + \sum_{j=1}^{n} v_j w_j a$ is a finite sum. Therefore *s* + *v* ∈ TT*a* and hence TT*a* is a subsemigroup of (T, +).

Now $uts = ut(\sum_{i=1}^{n} v_i w_i a)$

$= (\sum_{i=1}^{n} utv_i w_i a) \in$ TT*a*

$\Rightarrow$ TT*a* is a left ideal of T. Since T is a left simple ternary semigroup, TT*a* = T

Therefore TT*a* = T for all *a* $\in$T.

Conversely suppose that TT*a* = T for all *a* $\in$T.

Let L be a left ideal of T. Let *l* $\in$L. Then *l* $\in$T. By assumption TT*l* = T.

Let *t* $\in$T . Then *t* $\in$ TT*l*

$\Rightarrow t = \sum_{i=1}^{n} u_i v_i l$ for some u_i, v_i $\in$ T.

l $\in$ L; u_i, v_i $\in$ T and L is a left ideal of T $\Rightarrow \sum_{i=1}^{n} u_i v_i l$ $\in$ L $\Rightarrow$ t $\in$ L.

Therefore T $\subseteq$ L. Clearly L $\subseteq$ T and hence L = T.

Therefore T is the only left ideal of T. Hence T is left simple ternary semiring.

IV. LATERAL TERNARY IDEALS

We now introduce the term of a lateral ternary ideal in a ternary semiring.

Definition* IV.1**: A nonempty subset of a ternary semiring T is said to be a ***lateral ternary ideal or simply ***lateral ideal*** of T if

(1) $a, b \in$ A implies $a + b \in$ A.

(2) $b, c \in$ T , $a \in$ A implies $bac \in$ A.

***Note* IV.2:** A nonempty subset of A of a ternary semigroup T is a lateral ideal of T if and only if A is additive subsemigroup of T and TAT $\subseteq$ A.

***Example* IV.3:** In the ternary semiring Z^0, nZ^0 is a lateral ideal for any $n \in$ N.

***Theorem* IV.4: The nonempty intersection of any two lateral ideals of a ternary semiring T is a lateral ideal of T.**

***Theorem* IV.5: The nonempty intersection of any family of lateral ideals of a ternary semiring T is a lateral ideal of T.**

***Theorem* IV.6: The union of any two lateral ideals of a ternary semiring T is a lateral ideal of T.**

***Theorem* IV.7: The union of any family of lateral ideals of a ternary semiring T is a lateral ideal of T.**

We now introduce a maximal lateral ternary ideal and lateral ternary ideal generated by a subset of a ternary semiring.

Definition* IV.8**: An ideal A of a ternary semiring T is said to be a ***maximal lateral ternary ideal provided A is a proper lateral ideal of T and is not properly contained in any proper lateral ideal of T.

Definition* IV.9**: Let T be a ternary semiring and A be a non-empty subset of T. The smallest lateral ideal of T containing A is called ***lateral ternary ideal of T generated by A.

***Theorem* IV.10: The lateral ideal of a ternary semiring T generated by a non-empty subset A is the intersection of all lateral ideals of T containing A.**

We now introduce a principal lateral ternary ideal of a ternary semiring and characterize principal lateral ternary ideal.

Definition* IV.11**: A lateral ideal A of a ternary semiring T is said to be the ***principal lateral ternary ideal generated by a if A is a lateral ideal generated by $\{a\}$ for some a $\in$ T. It is denoted by M (a) (or) $< a >_m$.

***Theorem* IV.12: If T is a ternary semiring and *a* $\in$ T then**

M(*a*) = $\left\{ \sum_{i=1}^{n} r_i a t_i + \sum_{j=1}^{n} u_j v_j a p_j q_j + na : r_i, t_i, u_j v_j p_j q_j \in T,\ n \in z_0^{+} \right\}$. **Where**

Σ denotes a finite sum and z_0^{+} is the set of all positive integer with zero.

***Proof* :** Similar to proof of the theorem III.12.

***Note* IV.13**: if T is ternary semiring and a $\in$ T then

L(a) = $T^e a T^e + T^e T^e a T^e T^e + na$.

We now introduce a lateral simple ternary semiring and characterize lateral simple ternary semiring.

Definition* IV.14**: A ternary semiring T is said to be ***lateral simple ternary semiring if T is its only lateral ideal.

***Theorem* IV.15: A ternary semiring T is a lateral simple ternary semiring if and only if T*a*T = TT*a*TT = T for all *a* $\in$ T.**

Proof: Similar to III.15.

V. RIGHT TERNARY IDEALS

We now introduce the term of a right ternary ideal in a ternary semiring.

Definition* V.1**: A nonempty subset A of a ternary semigroup T is a ***right ternary ideal or simply ***right ideal*** of T if

(1) $a, b \in$ A implies $a + b \in$ A.

(2) $b, c \in$ T , $a \in$ A implies $abc \in$ A.

***Note* V.2:** A nonempty subset A of a ternary semigroup T is a right ideal of T if and only if A is additive subsemigroup of T and ATT $\subseteq$ A.

***Example* V.3:** In the ternary semiring Z^0, nZ^0 is a right ideal for any $n \in$ N.

***Theorem* V.4: The nonempty intersection of any two right ideals of a ternary semiring T is a right ideal of T.**

***Theorem* V.5: The nonempty intersection of any family of right ideals of a ternary semiring T is a right ideal of T.**

***Theorem* V.6: The union of any two right ideals of a ternary semiring T is a right ideal of T.**

***Theorem* V.7: The union of any family of right ideals of a ternary semiring T is a right ideal of T.**

We now introduce a maximal right ideal and right ideal generated by a subset of a ternary semiring.

Definition* V.8**: An ideal A of a ternary semiring T is said to be a ***maximal right ideal provided A is a proper right ideal of T and is not properly contained in any proper right ideal of T.

Definition* V.9:** Let T be a ternary semiring and A be a non-empty subset of T. The smallest right ideal of T containing A is called ***right ideal of T generated by A.

***Theorem* V.10: The right ideal of a ternary semigroup T generated by a nonempty subset A is the intersection of all right ideals of T containing A.**

We now introduce a principal right ideal of a ternary semiring and characterize principal right ideal.

Definition* V.11**: A right ideal A of a ternary semiring T is said to be a ***principal right ideal generated by a if A is a right ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by
R (a) (or) $< a >_r$.

***Theorem* V.12: If T is a ternary semiring and $a \in$ T then R(a) =**
$\left\{\sum_{i=1}^{n} ar_it_i + na : r_i, t_i \in T,\ n \in z_0^{+}\right\}$. **Where Σ denotes a finite sum and z_0^{+} is the set of all positive integer with zero.**

Proof: Similar to the proof of the theorem III.12.

***Note* V.13**: If T is a ternary semiring and $a \in$ T then R (a) = $a\ T^eT^e + na$.

We now introduce a right simple ternary semiring and characterize right simple ternary semiring.

Definition* V.14:** A ternary semiring T is said to be ***right simple ternary semiring if T is its only right ideal.

***Theorem* V.15: A ternary semiring T is a right simple ternary semiring if and only if aTT = T for all $a \in$ T.**

Proof: Similar to III.15.

VI. TWO SIDED TERNARY IDEALS

We now introduce the term of a two sided ternary ideal in a ternary semiring.

Definition* VI.1** : A nonempty subset A of a ternary semiring T is a ***two sided ternary ideal or simply ***two sided ideal*** of T if

(1) $a, b \in$ A implies $a + b \in$ A

(2) $b, c \in$ T , $a \in$ A implies $bca \in$ A, $abc \in$ A.

***Note* VI.2:** A nonempty subset A of a ternary semiring T is a two sided ideal of T if and only if it is both a left ideal and a right ideal of T.

***Example* VI.3:** In the ternary semiring Z^0, nZ^0 is a two sided ideal for any $n \in$ N.

***Theorem* VI.4: The nonempty intersection of any two sided ideals of a ternary semiring T is a two sided ideal of T.**

***Theorem* VI.5: The nonempty intersection of any family of two sided ideals of a ternary semiring T is a two sided ideal of T.**

***Theorem* VI.6: The union of any two two sided ideals of a ternary semiring T is a two sided ideal of T.**

***Theorem* VI.7: The union of any family of two sided ideals of a ternary semiring T is a two sided ideal of T.**

We now introduce a maximal two sided ideal and two sided ideal generated by a subset of a ternary semiring.

Definition* VI.8**: An ideal A of a ternary semiring T is said to be a ***maximal two sided ideal provided A is a proper two sided ideal of T and is not properly contained in any proper two sided ideal of T.

Definition* VI.9**: Let T be a ternary semiring and A be a non-empty subset of T. The smallest two sided ideal of T containing A is called ***two sided ideal of T generated by A.

***Theorem* VI.10: The two sided ideal of a ternary semiring T generated by a non-empty subset A is the intersection of all two sided ideals of T containing A.**

We now introduce a principal two sided ideal of a ternary semiring and characterize principal two sided ideal.

Definition* VI.11** : A two sided ideal A of a ternary semiring T is said to be the ***principal two sided ideal provided A is a two sided ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by T (a) (or) $< a >_t$.

***Theorem* VI.12: If T is a ternary semigroup and $a \in$ T then**

$$\mathbf{T}(a) = \left\{ \sum_{i=1}^{n} \begin{array}{c} r_is_ia + \sum_{j=1}^{n} at_ju_j + \sum_{k=1}^{n} l_km_kap_kq_k \\ +na : r_i, s_i, t_j, u_j, l_km_k, \\ p_k, q_k \in T \text{ and } n \in Z_0^{+} \end{array} \right\}.$$

Where Σ denotes a finite sum and z_0^{+} is the set of all positive integer with zero.

Proof: Similar to the proof of the theorem III.12.

***Note* VI.13:** if T is ternary semiring and $a \in$ T then T(a) = $T^eT^ea + aT^eT^e + T^eT^eaT^eT^e + na$.

Definition* VI.14:** A ternary semiring T is said to be a ***left duo ternary semiring provided every left ideal of T is a two sided ideal of T.

Definition* VI.15:** A ternary semiring T is said to be a ***right duo ternary semiring provided every right ideal of T is a two sided ideal of T.

Definition* VI.16:** A ternary semiring T is said to be a ***duo ternary semiring provided it is both a left duo ternary semiring and a right duo ternary semiring.

***Theorem* VI.17: A ternary semiring T is a duo ternary semiring if and only if $xT^eT^e = T^eT^ex$ for all $x \in$ T.**

Proof: Suppose that T is a duo ternary semiring and $x \in$ T.

Let $t \in xT^eT^e$. Then $t = \sum_{i=1}^{n} xu_iv_i$ for some $u_i, v_i \in T^e$.

Since T^eT^ex is a left ideal of T, T^eT^ex is a ideal of T.

So $x \in T^eT^ex$, $u_i, v_i \in$ T, T^eT^ex is a ideal of T $\Rightarrow \sum_{i=1}^{n} xu_iv_i \in T^eT^ex \Rightarrow t \in T^eT^ex$.

Therefore $xT^eT^e \subseteq T^eT^ex$. Similarly we can prove that $T^eT^ex \subseteq xT^eT^e$.

Therefore $xT^eT^e = T^eT^ex$ for all $x \in$ T.

Conversely suppose that $xT^eT^e = T^eT^ex$ for all $x \in$ T. Let A be a left ideal of T.

Let $x \in$ A, $u_i, v_i \in$ T.

Then $\sum_{i=1}^{n} xu_iv_i \in xT^eT^e = T^eT^ex$

$\Rightarrow \sum_{i=1}^{n} xu_iv_i = \sum_{i=1}^{n} s_it_ix$ for some $s_i, t_i \in T^e$.

Let $x \in$ A, $s_i, t_i \in$ T, A is a left ideal of T $\Rightarrow \sum_{i=1}^{n} s_it_ix \in$ A $\Rightarrow$

$\sum_{i=1}^{n} xu_iv_i \in$ A.

Therefore A is a right ideal of T and hence A is a ideal of T. Therefore T is left duo ternary semiring.
Similarly we can prove that T is a right duo ternary semiring.
Hence T is duo ternary semiring.

***Theorem* VI.18: Every commutative ternary semiring is a duo ternary semiring.**

***Proof*:** Suppose that T is a commutative ternary semiring. Therefore
$x\mathrm{T}^e\mathrm{T}^e = \mathrm{T}^e\mathrm{T}^e x$ for all $x \in$ T. By theorem VI.17, T is a duo ternary semiring.

***Theorem* VI.19: Every normal ternary semiring is a duo ternary semiring.**

***Proof*:** Suppose that T is normal ternary semiring. Then xyT = Txy for all $x, y \in$ T
$\Rightarrow x\mathrm{TT} = \mathrm{TT}x$ for all $x \in \mathrm{T} \Rightarrow x\mathrm{T}^e\mathrm{T}^e = \mathrm{T}^e\mathrm{T}^e x$ for all $x \in$ T. Therfore by theoem VI.17, T is a duo ternary semiring.

***Theorem* VI.20: Every quasi commutative ternary semiring is a duo ternary semiring.**

***Proof*:** Suppose that T is a quasi commutative ternary semiring. Then for each $a, b, c \in$ T, there exists a natural number n such that $abc = b^n ac = bca = c^n ba = cab = a^n cb$. Let A be a left ideal of T. Therefore TTA ⊆ A. Let $a \in$ A and $s, t \in$ T. Since T is a quasi commutative ternary semiring, there exist a natural number n such that $ast = t^n sa \in$ TTA ⊆ A. Therefore $ast \in$ A for all $a \in$ A and $s, t \in$ T and hence ATT ⊆ A. Thus A is right ideal of T. Therefore T is a left duo ternary semiring. Similarly we can prove that T is a right duo ternary semiring. Therefore every quasi commutative ternary semiring is a duo ternary semiring.

VII. TERNARY IDEALS

We now introduce the term ternary ideal of a ternary semiring.

Definition* VII.1**: A nonempty subset A of a ternary semiring T is said to be ***ternary ideal or simply an ***ideal*** of T if

(1) $a, b \in$ A implies $a + b \in$ A

(2) $b, c \in$ T, $a \in$ A implies $bca \in$ A, $bac \in$ A, $abc \in$ A.

***Note* VII.2**: A nonempty subset A of a ternary semigroup T is an ideal of T if and only if it is left ideal, lateral ideal and right ideal of T.

***Example* VII.3**: Let N be the set of all natural numbers. Define the ternary operation from $N \times N \times N \rightarrow$ N as $(a, b, c) = a.b.c$ where '.' is usual multiplication. Then N is a ternary semiring and A = 3N is an ideal of the ternary semiring N.

***Example* VII.4:** Consider the ternary semiring Z_0^- under usual addition and ternary multiplication, Let I = {0, -3} ∪ {-5, -6, -7, …..} is an ideal of Z_0^-.

***Theorem* VII.5: The nonempty intersection of any two ideals of a ternary semiring T is an ideal of T.**

***Theorem* VII.6: The nonempty intersection of any family of ideals of a ternary semiring T is an ideal of T.**

***Theorem* VII.7: The union of any two ideals of a ternary semiring T is a ternary ideal of T.**

***Theorem* VII.8: The union of any family of ideals of a ternary semiring T is an ideal of T.**

We now introduce a proper ideal, trivial ideal, maximal ideal and globally idempotent ideal of a ternary semiring and globally idempotent ternary semiring.

Definition* VII.9**: An ideal A of a ternary semiring T is said to be a ***proper ideal of T if A is different from T.

Definition* VII.10**: An ideal A of a ternary semiring T is said to be a ***trivial ideal provided T\ A is singleton.

Definition* VII.11**: An ideal A of a ternary semiring T is said to be a ***maximal ideal provided A is a proper ideal of T and is not properly contained in any proper ideal of T.

Definition* VII.12:** An ideal A of a ternary semiring T is called a ***globally idempotent ideal if A^n = A for all odd natural number n.

Definition* VII.13:** A ternary semigroup T is said to be a ***globally idempotent ternary semiring if T^n = T for all odd natural number n.

***Theorem* VII.14: If A is an ideal of a ternary semiring T with multiplicative identity *e* and *e* ∈ A then A = T.**

Proof: Clearly A ⊆ T. Let $t \in$ T. $e \in$ A, $t \in$ T, A is an ideal of T ⇒ $eet \in$ A ⇒ $t \in$ A
⇒ T ⊆ A. A ⊆ T, T ⊆ A ⟹ T = A.

***Theorem* VII.15: If T is a ternary semiring with multiplicative identity *e* then the union of all proper ideals of T is the unique maximal ideal of T.**

Proof: Let M be the union of all proper ideals of T. Since e is not an element of any proper ideal of T, $e \notin$ M. Therefore M is a proper subset of T. By theorem 7.8, M is a ternary ideal of T. Thus M is a proper ideal of T. Since M contains all proper ideals of T , M is a maximal ideal of T. If M_1 is any maximal ideal of T, then $M_1 \subseteq$ M ⊂ T and hence M_1 = M. Therefore M is the unique maximal ideal of T.

We now introducing ideal generated by a subset of a ternary semiring.

Definition* VII.16**: Let T be a ternary semiring and A be a non-empty subset of T. The smallest ideal of T containing A is called ***ideal of T generated by A.

***Theorem* VII.17: The ideal of a ternary semiring T generated by a non-empty subset A is the intersection of all ideals of T containing A.**

We now introduce a principal ideal and characterize principal ideal of a ternary semiring.

Definition* VII.18** : An ideal A of a ternary semiring T is said to be ***a principal ideal provided A is an ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by J (a) (or) < a >.

***Theorem* VII.19: If T is a ternary semiring and *a* ∈ T then**
J(*a*)

$$= \Big\{ \sum_{i=1}^{n} p_i q_i a + \sum_{j=1}^{n} a r_j s_j + \sum_{k=1}^{n} t_k a u_k + \sum_{l=1}^{n} v_l w_l a x_l y_l + na$$

$$: p_i, q_i, r_j, s_j, t_k, u_k, v_l, w_l, x_l y_l \in T,\ n \in Z_0^{+} \Big\}.$$

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Where Σ denotes a finite sum and z_0^{+} is the set of all positive integer with zero.

Proof: Let

A = $\left\{ \sum_{i=1}^{n} p_i q_i a + \sum_{j=1}^{n} a r_j s_j + \sum_{k=1}^{n} t_k a u_k + \sum_{l=1}^{n} v_l w_l a x_l y_l + na \right.$

$\left. : p_i, q_i, r_j, s_j, t_k, u_k, v_l, w_l, x_l y_l \in T,\ n \in Z_0^{+} \right\}.$

Let $s,\ t \in T$, $a,\ b \in$ A. $a,\ b \in$ A, then $a = \sum_{i=1}^{n} p_i q_i a + \sum_{j=1}^{n} a r_j s_j + \sum_{k=1}^{n} t_k a u_k + \sum_{l=1}^{n} v_l w_l a x_l y_l + na$

and

$b = \sum_{m=1}^{n} p_m q_m a + \sum_{o=1}^{n} a r_o s_o + \sum_{g=1}^{n} t_g a u_g + \sum_{h=1}^{n} v_h w_h a x_h y_h + na$

for $p_i, q_i, r_j, s_j, t_k, u_k, v_l, w_l, x_l, y_l,$ $p_m q_m, r_o, s_o, t_g, u_g, v_h, w_h, x_h,$ $y_h \in T,\ n \in Z_0^{+}$

Now $a + b$ =

$\sum_{i=1}^{n} p_i q_i a + \sum_{j=1}^{n} a r_j s_j + \sum_{k=1}^{n} t_k a u_k + \sum_{l=1}^{n} v_l w_l a x_l y_l + na$ +

$\sum_{m=1}^{n} p_m q_m a + \sum_{o=1}^{n} a r_o s_o + \sum_{g=1}^{n} t_g a u_g + \sum_{h=1}^{n} v_h w_h a x_h y_h + na$

is a finite sum.

Therefore $a + b \in$ A and hence A is a additive subsemigroup of T.

Let $r \in$ A. Then $r = \sum_{i=1}^{n} p_i q_i a + \sum_{j=1}^{n} a r_j s_j + \sum_{k=1}^{n} t_k a u_k + \sum_{l=1}^{n} v_l w_l a x_l y_l + na$

for $p_i, q_i, r_j, s_j, t_k, u_k, v_l, w_l, x_l, y_l \in T,\ n \in z_0^{+}$

If $r = \sum_{i=1}^{n} p_i q_i a + \sum_{j=1}^{n} a r_j s_j + \sum_{k=1}^{n} t_k a u_k + \sum_{l=1}^{n} v_l w_l a x_l y_l + na \in$ T. Therefore A ⊆ T and hence A is a smallest ideal containing a. Therefore

J(a) = A =

$\left\{ \sum_{i=1}^{n} p_i q_i a + \sum_{j=1}^{n} a r_j s_j + \sum_{k=1}^{n} t_k a u_k + \sum_{l=1}^{n} v_l w_l a x_l y_l + na \right.$

$\left. : p_i, q_i, r_j, s_j, t_k, u_k, v_l, w_l, x_l y_l \in T,\ n \in Z_0^{+} \right\}.$

***Note* VII.20**: If T is a ternary semigroup and $a \in$ T then J (a) = a + TTa + aTT + TaT + TTaTT = $T^e\ T^e\ a T^e\ T^e$.

***Theorem* VII.21: Let A, B be two ideals of ternary semiring T with zero and let A + B = { $a + b$: a ∈ A, b ∈ B}. Then A + B is an ideal generated by (A ∪ B).**

Proof: We first show that A + B is an ideal. Let $a_1 + b_1 \in$ A + B and $a_2 + b_2 \in$ A + B.

Let s, t be any arbitrary elements of T. Then, since A, B are ideals.

Then we have $a_1, a_2 \in$ A, $b_1, b_2 \in$ B

$\Rightarrow a_1 + a_2 \in$ A and $b_1 + b_2 \in$ B

$\Rightarrow (a_1 + a_2) + (b_1 + b_2) \in$ A + B $\Rightarrow (a_1 + b_1) + (a_2 + b_2) \in$ A + B.

Now $a_1, a_2 \in$ A, $b_1, b_2 \in$ B, $s, t \in$ T, we have sta_1, sa_1t, $a_1st \in$ A; stb_1, sb_1t, $b_1st \in$ B

$\Rightarrow sta_1 + stb_1 \in$ A + B, $sa_1t + sb_1t \in$ A + B, $a_1st + b_1st \in$ A + B

$\Rightarrow st(a_1 + b_1) \in$ A + B; $s(a_1 + b_1)t \in$ A + B; $(a_1 + b_1)st \in$ A + B.

Therefore A + B is an ideal of T. We now show that A + B is generated by A ∪ B.

Let $a \in$ A. Since $0 \in$ B, we have $a + 0 \in$ A + B. But $a + 0 = a$. Thus $a \in$ A $\Rightarrow a \in$ A + B.

Therefore A ⊆ A + B. Similarly B ⊆ A + B. It follows that A ∪ B ⊆ A + B. Thus A + B is an ideal containing A ∪ B. Also if any ideal contains A ⊂ B, then it is easy to see that it must contain A + B. Hence A + B is the smallest ideal containing A ∪ B,. i.e. A + B is the ideal generated by A ∪ B. In other words, A + B = (A ∪ B).

***Corollary* VII.22 : The left ideal generated by the union A ∪ B of two left ideals is the set A + B consisting of the elements of ternary semiring T obtained on adding any element of A to any element of B.**

***Theorem* VII.23: In any ternary semiring T, the following are equivalent.**

1) **Principal ideals of T form a chain.**
2) **Ideals of T form a chain.**

Proof: (1) $\Longrightarrow$ (2): Suppose that principal ideals of T form a chain.

Let A, B be two ideals of T. Suppose if possible A ⊈B , B ⊈ A.

Then there exist $a \in$ A\B and $b \in$ B\A

$a \in$ A $\Longrightarrow$ < a > ⊆ A and $b \in$ B $\Longrightarrow$ < b > ⊆ B.

Since principal ideals form a chain, either < a > ⊆ < b > or < b > ⊆ < a > .

If < a > ⊆ < b >, then $a \in$ < b > ⊆ B. It is a contradiction.

If < b > ⊆ < a >, then $b \in$ < a > ⊆ A. It is also a contradiction.

Therefore A $\subseteq$ B or B $\subseteq$ A and hence ideals form a chain.
(2) $\Rightarrow$ (1) : Suppose that ideals of T form a chain.
Then clearly principal ideals of T form a chain.
We now introduce a simple ternary semiring and characterize simple ternary semirings.

Definition* VII.24**: A ternary semigroup T is said to be ***simple ternary semiring if T is its only ideal of T.

***Theorem* VII.25: If T is a left simple ternary semiring or a lateral simple ternary semiring or a right simple ternary semiring then T is a simple ternary semiring.**

Proof: Suppose that T is a left simple ternary semiring. Then T is the only left ideal of T. If A is an ideal of T, then A is a left ideal of T and hence A = T.
Therefore T itself is the only ideal of T and hence T is a simple ternary semiring.
Suppose that T is a lateral simple ternary semiring. Then T is the only lateral ideal of T. If A is an ideal of T, then A is a lateral ideal of T and hence A = T.
Therefore T itself is the only ideal of T and hence T is a simple ternary semiring. Similarly if T is right simple ternary semiring then T is simple ternary semiring.

***Theorem* VII.26: A ternary semiring T is simple ternary semiring if and only if TT*a*TT = T for all $a \in$ T.**

Proof: Suppose that T is a simple ternary semiring and $a \in$ T. Let $s, t \in$ TTaTT; $t, u \in$ T

$s, t \in$ TTaTT $\Rightarrow s = \sum_{i=1}^{n} r_i s_i a v_i w_i$ and $t = \sum_{j=1}^{n} r_j s_j a v_j w_j$

where $r_i, r_j, s_i, s_j, v_i, v_j, w_i, w_j \in$ T, $n \in Z_0^+$. Therefore

$$s + t = \sum_{i=1}^{n} r_i s_i a v_i w_i + \sum_{j=1}^{n} r_j s_j a v_j w_j$$

is a finite sum and hence $s + t \in$ TTaTT. Therefore TTaTT is a subsemigroup of (T, +).

Now $sut = (\sum_{i=1}^{n} r_i s_i a v_i w_i)ut$

$= (\sum_{i=1}^{n} r_i s_i a u t v_i w_i) \in$ TTaTT and

$uts = ut(\sum_{i=1}^{n} r_i s_i a v_i w_i)$

$= \sum_{i=1}^{n} u t r_i s_i a v_i w_i \in TTaTT$

$\Rightarrow$ TTaTT is a ideal of T. Since T is a simple ternary semiring, TTaTT = T
Therefore TTaTT = T for all $a \in$ T.
Conversely suppose that TTaTT = T for all $a \in$ T.
Let I be an ideal of T. Let $l \in$ I. Then $l \in$ T . By assumption TTlTT = T.
Let $t \in$ T . Then $t \in$ TTlTT

$\Rightarrow t = \sum_{i=1}^{n} r_i s_i l u_i v_i$ for some $r_i, s_i, u_i, v_i \in$ T.

$l \in$ I; $r_i, s_i, u_i, v_i \in$ T and I is an ideal of T $\Rightarrow \sum_{i=1}^{n} r_i s_i l u_i v_i \in$ I $\Rightarrow t \in$ R.

Therefore T $\subseteq$ I. Clearly I $\subseteq$ T and hence I = T. Therefore T is the only ideal of T. Hence T is a simple ternary semiring.

***Theorem* VII.27: A ternary semiring T is regular then every principal ideal of T is generated by an idempotent.**

Proof: Suppose T is a regular ternary semiring. Let $< a >$ be a principal ideal of T. Since T is regular, $\exists\, x, y \in$ T $\ni axaya = a$.
Now $(axay)^3 = axayaxayaxay = axayaxay = axay$. Let $axay = e$.
$a = axaya = axayaxaya = eea \in < e > \Rightarrow < a > \subseteq < e >$.
Now $e = axay \in < a > \Rightarrow < e > \subseteq < a >$. Therefore $< a > = < e >$ and hence every principal ideal generated by an idempotent.

We now introduce a semi simple element of a ternary semigroup and a semi simple ternary semiring.

Definition* VII.28**: An element a of a ternary semiring T is said to be ***semi simple if $a \in < a >^3$ i.e. $< a >^3 = <a>$.

Theorem* VII.29:** An element a of a ternary semiring T is said to be ***semi simple if $a \in < a >^n$ i.e. $< a >^n = <a>$ for all odd natural number n.

Proof: Suppose that a is semi simple element of T. Then $< a >^3 = < a >$.
Let $a \in$ T and n is odd natural number.
If $n = 1$ then clearly $a \in < a >$.
If $n = 3$ and a is semi simple then $< a >^3 = < a >$.
If $n = 5$ then $< a >^5 = < a >^3 < a >^2 = < a > < a >^2 = < a >^3 = < a >$.
Therefore by induction of n is an odd natural number, we have $< a >^n = < a >$.
The converse part is trivial.

Definition* VII.30** : A ternary semiring T is called ***semi simple ternary semiring provided every element in T is semi simple.

***Theorem*VII.31: Let T be a ternary semiring and $a \in T$. If *a* is regular then *a* is semi simple.**

Proof: Suppose that a is regular. Then $a = axaya$ for some $x, y \in$ T $\Rightarrow a \in <a>^3$.
Therefore a is semi simple.

***Theorem* VII.32: Let *a* be an element of a ternary semiring T. If *a* is left regular or lateral regular or right regular, then *a* is semi simple.**

Proof: Suppose a is left regular. Then $a = a^3xy$ for some x, $y \in$ S $\Rightarrow a \in < a >^3$.
Therefore a is semi simple.
If a is right regular, then $a = xya^3$ for some $x, y \in$ S $\Rightarrow a \in < a >^3$.
Therefore a is semi simple.

***Theorem* VII.33: Let *a* be an element of a ternary semiring T. If *a* is intraregular then *a* is semi simple.**

Proof: Suppose a is intra regular. Then $a = xa^5y = xa^2a^3y$ for some $x, y \in$ T
$\Rightarrow a \in < a >^3$. Therefore a is semi simple.

VIII. CONCLUSION

In this paper mainly we studied about structure of certain ideals in ternary semirings.

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