General Pattern of Total Coloring of a Prism Graph of n-Layers and a Grid Graph

S. Sudha, K. Manikandan

***Abstract:** Behzad [1] introduced the total coloring of a graph G as an assignment of colors to the elements (vertices and edges) of G such that no two elements receive the same color. In this paper, we have obtained the total coloring of prism graph of n-layers and grid graphs in general using five colors only and found that the total-chromatic number is $\Delta(G)+1=5$ for both prism graph of n-layers and grid graphs. AMS Subject Classification: 05C15*

***Keywords:** Total coloring, prism graph of n-layers, grid graphs.*

I. INTRODUCTION

Behzad [1] has defined the total coloring of a graph. A graph G is said to admit total coloring if
(i) no two adjacent vertices have the same color
(ii) no two adjacent edges have the same color
(iii) no edge and its end vertices are assigned with the same color.

The minimum number of colors required for a graph G in this way, is its chromatic number and it is denoted by $\chi_t(G)$ Behzad [1] discussed about the total coloring of paths, cycles, complete graphs and complete bipartite graphs. Behzad [1] and Vizing [2] has given the total coloring conjecture as $\Delta(G)+1 \leq \chi_t(G) \leq \Delta(G)+2$ for a simple graph G. Jensen et.al [3] and Borodin [4] have also discussed about the total coloring of graphs. Gutman [5] gave the total coloring of thorny graphs. Heckman et.al [6] discussed the circular total coloring of graphs. Sudha et.al [7,8] have given the total coloring and (k,d)-total coloring for prisms Y_n and $S(n,m)$-graphs (Sudha graphs). The prism graph Y_m^n is the cartesian product of the cycle C_m with the path P_n.

DEFINITION 1: A prism graph of n-layers, Y_m^n is a simple graph given by the cartesian product of the cycle C_m and the path P_n. This graph consists of mn vertices and $m(2n-1)$ edges. The number of circles in Y_m^n is n which depends on the length of path, P_n and we say that the prism is of n-layers.

ILLUSTRATION 1: Consider the prism graph $C_4 \times P_3$. Let the vertex sets of the cycle C_4 and the path P_3 be $\{u_1,u_2,u_3,u_4\}$ and $\{w_1,w_2,w_3\}$ respectively. By the above definition, we obtain the prism graph of 3-layers with the vertices denoted by $v_{i,j}$ for $i=1,2,3$ and $j=1,2,3,4$ respectively.

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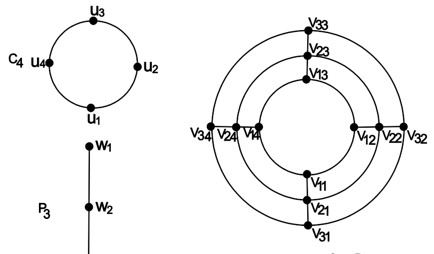
Figure 1. $C_4 \times P_3$

DEFINITION 2: A grid graph is defined as the cartesian product of two paths P_m and P_n.

The graph $P_m \times P_n$ has vertices denoted by $v_{i,j}$ and the number of edges in $P_m \times P_n$ is *2mn-m-n.*

The vertex set and the edge set of $P_m \times P_n$ are given as $V(P_m \times P_n)=\{(g,h)/g \,\epsilon V(G)$ and $h \,\epsilon\, V(H)\}$ and $E(P_m \times P_n)=\{(g,h)\,(g',h')/g=g'$ and $hh'\epsilon E(H)$ or $gg'\epsilon E(G)$ and $h=h'\}$.

ILLUSTRATION 2: Consider the grid graph $P_3 \times P_4$. Let the vertex sets of the paths P_3 and P_4 be $\{u_1,u_2,u_3\}$ and $\{w_1,w_2,w_3,w_4\}$ respectively. By the above definition of grid graphs, we obtain the cartesian product $P_3 \times P_4$ with the vertices denoted by $v_{i,j}$ for $i=1,2,3$ and $j=1,2,3,4$ respectively.

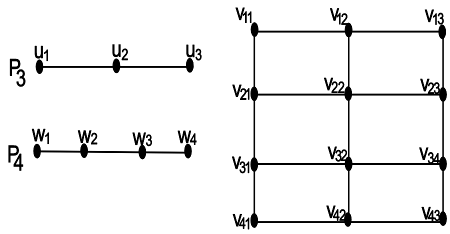
Figure 2. $P_3 X P_4$

In this paper, we have assigned the colors for both vertices and edges of the prism graph of n-layers as well as grid graphs in such a way that it satisfies the definition of total coloring and its total-chromatic number is found to be $\Delta(G)+1=5$ for both the graphs.

II. TOTAL COLORING OF PRISM GRAPH OF n-LAYERS

Theorem 1. The prism graph of n-layers, Y_m^n $(m \geq 3$, $n \geq 3)$, admits total coloring and its total-chromatic number is 5.

Proof: Consider the prism graph Y_m^n $(m \geq 3, n \geq 3)$, which has n-layers with m vertices on each layer.

Let the vertex set of the prism graph of n-layers Y_m^n is given by

$$V(Y_m^n)=\bigcup_{i=1}^{n}(v_{i,j}\ ;\ 1 \leq j \leq m),$$

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where i stands for the layer and j is its vertex on the i^{th} layer.

and its edge set is given by

$$E(Y_m^n) = \bigcup_{i=1}^{n} (v_{i,j}v_{i,j+1};\ 1 \le j \le m)$$

$$\cup \bigcup_{i=1}^{n} (v_{i,j}v_{i+1,j}\ ;\ 1 \le j \le m),$$

Where $v_{i,j}v_{i,j+1}$ represent an edge between $v_{i,j}$ and $v_{i,j+1}$ on the i^{th} layer; and $v_{i,j}v_{i+1,j}$ represent an edge joining the j^{th} vertices of the layers i and $i+1$.

The prism graph Y_m^n can be colored totally with five colors, say $\{1,2,3,4,5\}$ as follows:

In coloring we use the notations S_1, S_2, S_3, S_4 and S_5 for the set of colors $\{1,2,3,4,5\}$, $\{2,3,4,5,1\}$, $\{3,4,5,1,2\}$, $\{4,5,1,2,3,\}$ and $\{5,1,2,3,4\}$ respectively.

There are five cases depending on the value of m.

(i) $m \equiv 0(mod\ 5)$

(ii) $m \equiv 1(mod\ 5)$

(iii) $m \equiv 2(mod\ 5)$

(iv) $m \equiv 3(mod\ 5)$

(v) $m \equiv 4(mod\ 5)$

Case $(\boldsymbol{i})$: Let $m \equiv 0\ (mod\ 5)$. The value of n is arbitrary.

For $1 \le i \le n\ , 1 \le j \le m$

Define the function f_1 from the vertices of Y_m^n to the color set $\{1,2,3,4,5\}$ as follows:

$$f_1(v_{i,j}) = \begin{cases} S_1^\alpha, & i \equiv 1(mod\ 5) \\ S_3^\alpha, & i \equiv 2(mod\ 5) \\ S_5^\alpha, & i \equiv 3(mod\ 5) \\ S_2^\alpha, & i \equiv 4(mod\ 5) \\ S_4^\alpha, & i \equiv 0(mod\ 5) \end{cases}$$

where the superscript α defines the number of times the S_s' to be repeated according to the value $\alpha = \frac{m}{5}$.

Define the function f_2 from the edges on the layers of the function to the color set $\{1,2,3,4,5\}$ as follows:

$$f_2(v_{i,j}v_{i,j+1}) = \begin{cases} S_3^\alpha, & i \equiv 1(mod\ 5) \\ S_1^\alpha, & i \equiv 2(mod\ 5) \\ S_4^\alpha, & i \equiv 3(mod\ 5) \\ S_5^\alpha, & i \equiv 4(mod\ 5) \\ S_2^\alpha, & i \equiv 0(mod\ 5) \end{cases}$$

Define the function f_3 from the edges joining the vertices of adjacent layers to the color set $\{1,2,3,4,5\}$ as follows:

$$f_3(v_{i,j}v_{i+1,j}) = \begin{cases} S_4^\alpha, & i \equiv 1(mod\ 5) \\ S_2^\alpha, & i \equiv 2(mod\ 5) \\ S_1^\alpha, & i \equiv 3(mod\ 5) \\ S_3^\alpha, & i \equiv 4(mod\ 5) \\ S_5^\alpha, & i \equiv 0(mod\ 5) \end{cases}$$

By using the above defined coloring pattern, the graph (Y_m^n) is total colored and the total-chromatic number of the prism graph of n-layers, $\chi_t(Y_m^n)$ is 5.

Case $(\boldsymbol{ii})$: Let $m \equiv 1\ (mod\ 5)$. The value of n is arbitrary.

For $1 \le i \le n\ , 1 \le j \le m$

Define the function f_1 from the vertices of Y_m^n to the color set $\{1,2,3,4,5\}$ as follows:

$$f_1(v_{i,j}) = \begin{cases} S_1^\alpha 123143, & i \equiv 1(mod\ 5) \\ S_3^\alpha 345252, & i \equiv 2(mod\ 5) \\ S_5^\alpha 512314, & i \equiv 3(mod\ 5) \\ S_2^\alpha 234145, & i \equiv 4(mod\ 5) \\ S_4^\alpha 451532, & i \equiv 0(mod\ 5) \end{cases}$$

where the superscript α defines the number of times the S_s' to be repeated according to the value $\alpha = \frac{m-6}{5}$.

Define the function f_2 from the edges on the layers of the function to the color set $\{1,2,3,4,5\}$ as follows:

$$f_2(v_{i,j}v_{i,j+1}) = \begin{cases} S_3^\alpha 345312, & i \equiv 1(mod\ 5) \\ S_1^\alpha 123135, & i \equiv 2(mod\ 5) \\ S_4^\alpha 451253, & i \equiv 3(mod\ 5) \\ S_5^\alpha 512514, & i \equiv 4(mod\ 5) \\ S_2^\alpha 234141, & i \equiv 0(mod\ 5) \end{cases}$$

Define the function f_3 from the edges joining the vertices of adjacent layers to the color set $\{1,2,3,4,5\}$ as follows:

$$f_3(v_{i,j}v_{i+1,j}) = \begin{cases} S_4^\alpha 451424, & i \equiv 1(mod\ 5) \\ S_2^\alpha 234541, & i \equiv 2(mod\ 5) \\ S_1^\alpha 123432, & i \equiv 3(mod\ 5) \\ S_3^\alpha 345323, & i \equiv 4(mod\ 5) \\ S_5^\alpha 512255, & i \equiv 0(mod\ 5) \end{cases}$$

By using the above defined coloring pattern, the graph Y_m^n is total colored and the total-chromatic number of the prism graph of n-layers, $\chi_t(Y_m^n)$ is 5.

Case $(\boldsymbol{iii})$: Let $m \equiv 2\ (mod\ 5)$. The value of n is arbitrary.

For $1 \le i \le n\ , 1 \le j \le m$

Define the function f_1 from the vertices of Y_m^n to the color set $\{1,2,3,4,5\}$ as follows:

$$f_1(v_{i,j}) = \begin{cases} S_1^\alpha 1234543, i \equiv 1(mod\ 5) \\ S_3^\alpha 3451351, i \equiv 2(mod\ 5) \\ S_5^\alpha 5123124, i \equiv 3(mod\ 5) \\ S_2^\alpha 2345245, i \equiv 4(mod\ 5) \\ S_4^\alpha 4512312, i \equiv 0(mod\ 5) \end{cases}$$

where the superscript α defines the number of times the S_s' to be repeated according to the value $\alpha = \frac{m-7}{5}$.

Define the function f_2 from the edges on the layers of the function to the color set $\{1,2,3,4,5\}$ as follows:

$$f_2(v_{i,j}v_{i,j+1}) = \begin{cases} S_3^\alpha 3451312, & i \equiv 1(mod\ 5) \\ S_1^\alpha 1234135, & i \equiv 2(mod\ 5) \\ S_4^\alpha 4512353, & i \equiv 3(mod\ 5) \\ S_5^\alpha 5123524, & i \equiv 4(mod\ 5) \\ S_2^\alpha 2345241, & i \equiv 0(mod\ 5) \end{cases}$$

Define the function f_3 from the edges joining the vertices of adjacent layers to the color set $\{1,2,3,4,5\}$ as follows:

$$f_3(v_{i,j}v_{i+1,j}) = \begin{cases} S_4^\alpha 4512224, & i \equiv 1(mod\ 5) \\ S_2^\alpha 2345542, & i \equiv 2(mod\ 5) \\ S_1^\alpha 1234411, & i \equiv 3(mod\ 5) \\ S_3^\alpha 3451133, & i \equiv 4(mod\ 5) \\ S_5^\alpha 5123455, & i \equiv 0(mod\ 5) \end{cases}$$

By using the above defined coloring pattern, the graph Y_m^n is total colored and the total-chromatic number of the prism graph of n-layers, $\chi_t(Y_m^n)$ is 5.

Case $(\boldsymbol{iv})$: Let $m \equiv 3\ (mod\ 5)$. The value of n is arbitrary.

For $1 \le i \le n\ , 1 \le j \le m$

Define the function f_1 from the vertices of Y_m^n to the color set $\{1,2,3,4,5\}$ as follows:

$$f_1(v_{i,j}) = \begin{cases} S_1^{\alpha}123, & i \equiv 1(mod\ 5) \\ S_3^{\alpha}351, & i \equiv 2(mod\ 5) \\ S_5^{\alpha}514, & i \equiv 3(mod\ 5) \\ S_2^{\alpha}245, & i \equiv 4(mod\ 5) \\ S_4^{\alpha}432, & i \equiv 0(mod\ 5) \end{cases}$$

where the superscript α defines the number of times the $S_s^{'}$ to be repeated according to the value $\alpha = \frac{m-3}{5}$.

Define the function f_2 from the edges on the layers of the function to the color set $\{1,2,3,4,5\}$ as follows:

$$f_2(v_{i,j}v_{i,j+1}) = \begin{cases} S_3^{\alpha}312, & i \equiv 1(mod\ 5) \\ S_1^{\alpha}135, & i \equiv 2(mod\ 5) \\ S_4^{\alpha}453, & i \equiv 3(mod\ 5) \\ S_5^{\alpha}524, & i \equiv 4(mod\ 5) \\ S_2^{\alpha}241, & i \equiv 0(mod\ 5) \end{cases}$$

Define the function f_3 from the edges joining the vertices of adjacent layers to the color set $\{1,2,3,4,5\}$ as follows:

$$f_3(v_{i,j}v_{i+1,j}) = \begin{cases} S_4^{\alpha}444, & i \equiv 1(mod\ 5) \\ S_2^{\alpha}222, & i \equiv 2(mod\ 5) \\ S_1^{\alpha}131, & i \equiv 3(mod\ 5) \\ S_3^{\alpha}313, & i \equiv 4(mod\ 5) \\ S_5^{\alpha}555, & i \equiv 0(mod\ 5) \end{cases}$$

By using the above defined coloring pattern, the graph Y_m^n is total colored and the total-chromatic number of the prism graph of n-layers, $\chi_t(Y_m^n)$ is 5.

Case (v): Let $m \equiv 4\ (mod\ 5)$. The value of n is arbitrary. For $1 \le i \le n, 1 \le j \le m$

Define the function f_1 from the vertices of Y_m^n to the color set $\{1,2,3,4,5\}$ as follows:

$$f_1(v_{i,j}) = \begin{cases} S_1^{\alpha}1234, & i \equiv 1(mod\ 5) \\ S_3^{\alpha}3452, & i \equiv 2(mod\ 5) \\ S_5^{\alpha}5131, & i \equiv 3(mod\ 5) \\ S_2^{\alpha}2423, & i \equiv 4(mod\ 5) \\ S_4^{\alpha}4515, & i \equiv 0(mod\ 5) \end{cases}$$

where the superscript α defines the number of times the $S_s^{'}$ to be repeated according to the value $\alpha = \frac{m-4}{5}$.

Define the function f_2 from the edges on the layers of the function to the color set $\{1,2,3,4,5\}$ as follows:

$$f_2(v_{i,j}v_{i,j+1}) = \begin{cases} S_3^{\alpha}3152, & i \equiv 1(mod\ 5) \\ S_1^{\alpha}1235, & i \equiv 2(mod\ 5) \\ S_4^{\alpha}4523, & i \equiv 3(mod\ 5) \\ S_5^{\alpha}5314, & i \equiv 4(mod\ 5) \\ S_2^{\alpha}2341, & i \equiv 0(mod\ 5) \end{cases}$$

Define the function f_3 from the edges joining the vertices of adjacent layers to the color set $\{1,2,3,4,5\}$ as follows:

$$f_3(v_{i,j}v_{i+1,j}) = \begin{cases} S_4^{\alpha}4541, & i \equiv 1(mod\ 5) \\ S_2^{\alpha}2314, & i \equiv 2(mod\ 5) \\ S_1^{\alpha}1245, & i \equiv 3(mod\ 5) \\ S_3^{\alpha}3152, & i \equiv 4(mod\ 5) \\ S_5^{\alpha}5423, & i \equiv 0(mod\ 5) \end{cases}$$

By using the above defined coloring pattern, the graph Y_m^n is total colored and the total-chromatic number of the prism graph of n-layers, $\chi_t(Y_m^n)$ is 5.

Illustration 3: Consider a prism graph of 5-layers, Y_5^5.

This illustration comes under case 1.

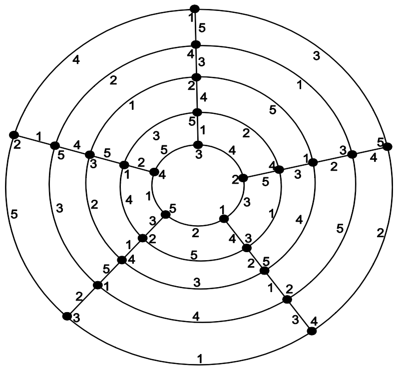

Figure 3. Prism Y_5^5

Hence by using the general color pattern of the above theorem, we assign the colors to the vertices and the edges as shown in the fig. 3. The total chromatic number is $\chi_t(G) = 5$.

Illustration 4: Consider a prism graph of 6-layers, Y_6^6.

This illustration comes under case 2.

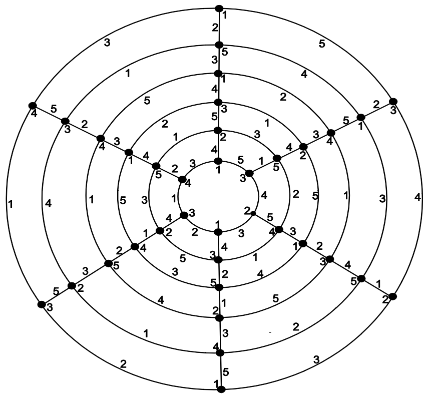

Figure 4. prism Y_6^6

Hence by using the general color pattern of the above theorem, we assign the colors to the vertices and the edges as shown in the fig 4. The total chromatic number is $\chi_t(G) = 5$.

Illustration 5: Consider a prism graph of 5-layers, Y_7^5.

This illustration comes under case 3.

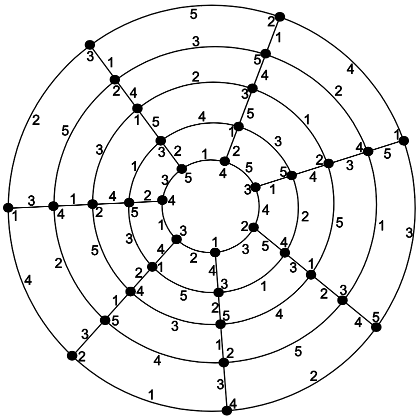

Figure 5. Prism Y_7^5

Hence by using the general color pattern of the above theorem, we assign the colors to the vertices and the edges as shown in the fig. 5. The total chromatic number is $\chi_t(G) = 5$.

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Illustration 6: Consider a prism graph of 4-layers, Y_3^4.

This illustration comes under case 4.

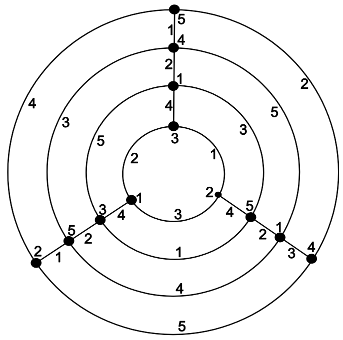

Figure 6. Prism Y_3^4

Hence by using the general color pattern of the above theorem, we assign the colors to the vertices and the edges as shown in the fig. 6. The total chromatic number is $\chi_t(G) = 5$.

Illustration 7: Consider a prism graph of 5-layers, Y_4^5.

This illustration comes under case 5.

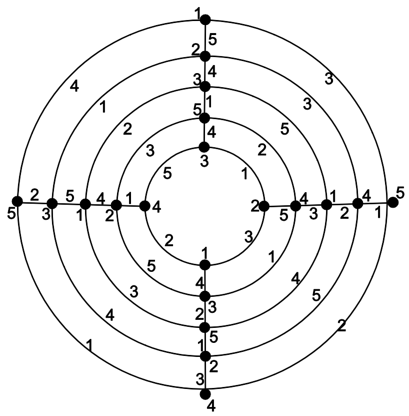

Figure 7. Prism Y_4^5

Hence by using the general color pattern of the above theorem, we assign the colors to the vertices and the edges as shown in the fig. 7. The total chromatic number is $\chi_t(G) = 5$.

III. TOTAL COLORING OF GRID GRAPHS

Theorem 2: The grid graph $P_m \times P_n$ $(m, n \geq 3)$ admits total coloring and its total chromatic number is 5.

Proof: If the vertex sets of the paths P_m and P_n are $\{u_1, u_2, \ldots, u_m\}$ and $\{w_1, w_2, \ldots, w_n\}$ respectively, then the vertex set of the grid graph $P_m \times P_n$ is given by

$$V(P_m \times P_n) = \bigcup_{i=1}^{m} (v_{i,j}\ ;\ 1 \leq j \leq n)$$

and its edge set is given by

$$E(P_m \times P_n) = \bigcup_{i=1}^{m} (v_{i,j} v_{i,j+1}\ , 1 \leq j \leq n) \cup \bigcup_{i=1}^{m} (v_{i,j} v_{i+1,j}\ \ , 1 \leq j \leq n)$$

For $1 \leq i \leq m\ , 1 \leq j \leq n$

Define the function f_1 from the vertices of the grid graph $P_m \times P_n$ to the color set $\{1,2,3,4,5\}$ as follows:

$$f_1(v_{i,j}) \equiv \begin{cases} (i+j)\ (mod\ 5), \text{if } (i+j) \not\equiv 0(mod\ 5) \\ 5(mod\ 5), \text{if } (i+j) \equiv 0(mod\ 5) \end{cases}$$

Define the function f_2 from the edges of the grid graph $P_m \times P_n$ to the color set $\{1,2,3,4,5\}$ as follows:

$$f_2(v_{i,j} v_{i,j+1}) \equiv \begin{cases} (i+j-1)\ \ (mod\ 5), \text{if } (i+j-1) \not\equiv 0(mod\ 5) \\ 5(mod\ 5), \text{if } (i+j-1) \equiv 0(mod\ 5) \end{cases}$$

Define the function f_3 from the edges of the grid graph $P_m \times P_n$ to the color set $\{1,2,3,4,5\}$ as follows:

$$f_3(v_{i,j} v_{i+1,j}) \equiv \begin{cases} (i+j+2)\ \ (mod\ 5), \text{if } (i+j+2) \not\equiv 0(mod\ 5) \\ 5(mod\ 5), \text{if } (i+j+2) \equiv 0(mod\ 5) \end{cases}$$

By using the above pattern of coloring, the grid graph can be given total colored and the total-chromatic number of the grid graph, $\chi_{tc}(P_m \times P_n)$is 5, for $m, n \geq 3$.

Illustration 8: Consider the grid graph $P_8 \times P_5$.

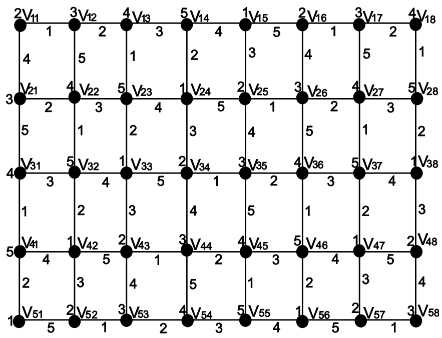

Figure 8. $P_8 \times P_5$

Hence by using the coloring pattern given in the above theorem, we assign the colors to the vertices and the edges as shown in the fig. 8. The total chromatic number is $\chi_t(G) = 5$.

IV. CONCLUSION

The total coloring of a prism graph of n-layers and the grid graph were discussed and found that the total coloring chromatic number of both them to be 5.

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