Special Pairs of Pythagorean Triangle

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Abstract— we illustrate the different methods of obtaining pairs of Pythagorean triangles which are such that, in each pair, the sum of the product of their generators is a perfect square. Also a few interesting properties among the pairs of Pythagorean triangles and special polygonal numbers are exhibited.

Keywords: Pair of Pythagorean triangles, special polygonal numbers

I. INTRODUCTION

The fascinating branch of mathematics is the theory of numbers where in Pythagorean triangles have been a matter of interest to various mathematicians and to the lovers of mathematics, because it is a treasure house in which the search for many hidden connection is a treasure hunt. For a rich variety of fascinating problems one may refer [1-17].A careful observer of patterns may note that there is a one to one correspondence between the polygonal numbers and the number of sides of the polygon.

In [18-20], special Pythagorean triangles connected with polygonal numbers and Nasty numbers are obtained. In [21] pairs of distinct Pythagorean triangles such that in each pair the difference between their perimeter is represented by i) $k\alpha^2$ ii) $k\alpha^n, n > 2$ iii) $3p_{2N}^3$ iv) $12pt_{2N}$ are obtained.

In this communication, we present different methods of obtaining pairs of Pythagorean triangles which are such that, in each pair, the sum of the product of their generators is a perfect square. A few interesting properties among the pairs of Pythagorean triangles and special polygonal numbers are exhibited.

II. NOTATION USED

- $t_{m,n}$ - Polygonal number of rank n with size m.
- P_n^m - Pyramidal number of rank n with size m.
- gn_a - Gnomonic number of rank a
- J_n - Jacobsthal number of rank n
- j_n - Jacobsthal-Lucas number of rank n
- Pt_n - Pantatope number of rank n
- Ky_n - Kyneya number of rank n
- pr_n - Pronic number of rank n

III. METHOD OF ANALYSIS

Let $T_1(x_1, y_1, z_1)$ and $T_2(x_2, y_2, z_2)$ be two distinct Pythagorean triangles, where

$$x_1 = m^2 - n^2,\ y_1 = 2mn,\ z_1 = m^2 + n^2 \text{ and} \quad (1)$$
$$x_2 = p^2 - q^2,\ y_2 = 2pq,\ z_2 = p^2 + q^2$$

$m, n (m > n > 0)$ and $p, q (p > q > 0)$ are the generators of T_1 and T_2 respectively.

We illustrate below the process of obtaining pairs of Pythagorean triangles such that the sum of the product of the generators is a perfect square.

$$\text{ie, } mn + pq = \alpha^2 \quad (2)$$

Different choices of T_1 and T_2 are obtained by solving (2) through employing different forms of linear transformations for the generators m, n, p and q in (2).

CHOICE: 1

Introducing the linear transformations

$$m = u + v, n = u - v, p = r + u, q = r - u, u \neq v \neq r \quad (3)$$

in (2), it gives

$$r^2 = \alpha^2 + v^2 \quad (4)$$

which is the well known Pythagorean equation which is satisfied by

$$v = 2ab, \alpha = a^2 - b^2, r = a^2 + b^2, a > b > 0$$

Substituting the values of v and r in (3) and using (1), the corresponding sides of T_1 and T_2 are as follows

$$x_1(a,b,u) = 8abu$$
$$y_1(a,b,u) = 2(u^2 - 4a^2b^2)$$
$$z_1(a,b,u) = 2(u^2 + 4a^2b^2)$$
$$x_2(a,b,u) = 4u(a^2 + b^2)$$
$$y_2(a,b,u) = 2[(a^2 + b^2)^2 - u^2]$$
$$z_2(a,b,u) = 2[(a^2 + b^2)^2 + u^2]$$

Properties:

- $x_1(a,b,u(u+1)) + x_2(a,b,u(u+1)) = 8t_{3,u} * t_{4,a+b}$
- $2Pr_a\, x_2(a,a+1,u) - gn_a x_1(a,a+1,u) \equiv 0 (\bmod 2a^2)$
- $3\{y_1(a,b,u) + z_1(a,b,u) + z_2(a,b,u) - y_2(a,b,u)\}$ is a Nasty number.
- $x_1(a,a,(a+1)) - y_1(a,a,(a+1)) + z_1(a,a,(a+1)) - 16P_a^5$ is a biquadratic integer.
- $x_1(a,(a+1)(a+2),a+3) + y_1(a,(a+1)(a+2),a+3) + z_1(a,(a+1)(a+2),a+3) = 192Pt_a + 4t_{4,a+3}$

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Note:1

The substitution of (4) may also be taken as

$v = a^2 - b^2, \alpha = 2ab, r = a^2 + b^2, a > b > 0$

For this choice, the corresponding sides of T_1 and T_2 are respectively as follows

$x_1(a,b,u) = 4u(a^2 - b^2)$

$y_1(a,b,u) = 2[u^2 - (a^2 - b^2)^2]$

$z_1(a,b,u) = 2[u^2 + (a^2 - b^2)^2]$

$x_2(a,b,u) = 4u(a^2 + b^2)$

$y_2(a,b,u) = 2[(a^2 + b^2)^2 - u^2]$

$z_2(a,b,u) = 2[(a^2 + b^2)^2 + u^2]$

Properties:

- $x_2(a,b,b+1) - x_1(a,b,b+1) = 16P_a^5$
- $y_1(2^a + 1,1,u) + y_2(2^a + 1,1,u) = 8(Ky_a + 2)$
- $3(J_{2n})^2 x_2(2^n,1,1) = (j_{2n})^2 x_1(2^n,1,1)$
- $y_1(2^n,1,u(u+1)) + z_1(2^n,1,u(u+1)) + y_2(2^n,1,u(u+1)) + z_2(2^n,1,u(u+1)) = 4(Pr_a)^2 + 4(j_{2n})^2$

CHOICE: 2

Introduction of the linear transformations

$m = 2v + 2s, n = 2s, p = r + 2s, q = r - 2s, r > 2s > 0$ (5)

in (2) leads to

$\alpha^2 - r^2 = 4vs$ (6)

which is satisfied by $\alpha = v + s, r = v - s$

Substituting the value r in (5) and using (1), the corresponding sides of T_1 and T_2 are as follows

$x_1(v,s) = 4v(v + 2s)$

$y_1(v,s) = 8s(v + s)$

$z_1(v,s) = 4(v^2 + 2s^2 + 2vs)$

$x_2(v,s) = -8s^2 + 8vs$

$y_2(v,s) = 2(v^2 - 2vs - 3s^2)$

$z_2(v,s) = 2(5s^2 + v^2 - 2vs)$

Properties:

- $x_1(t_{3,v}, t_{3,v+2}) + y_1(t_{3,v}, t_{3,v+2}) = 384Pt_v + 16t_{3,v} + 8(Pr_{a+2})^2$
- $y_1(2v^2 - 1, v) + x_2(2v^2 - 1, v) = 16SO_v$
- $x_2(v, s(s+1)(s+2)) + y_2(v, s(s+1)(s+2)) + z_2(v, s(s+1)(s+2)) + 24P_s^3 \equiv 0 \pmod 4$
- $6\{y_1(v,v) - x_1(v,v)\}$ is a Nasty number.

Note:2

(6) is equivalent to the following systems of equations

i) $\alpha + r = 2vs$, $\alpha - r = 2$ ii) $\alpha + r = vs$, $\alpha - r = 4$

Now solving the system of equations (i), we have

$\alpha = vs + 1, r = vs - 1,$

Using the value of r in (5) and in view of (1), the sides of T_1 and T_2 are given by

$x_1(v,s) = 4v^2 + 8vs$

$y_1(v,s) = 8(vs + s^2)$

$z_1(v,s) = 4(v^2 + 2s^2 + 2vs)$

$x_2(v,s) = 8vs^2 - 8s$

$y_2(v,s) = 2[(vs)^2 - 2vs - 4s^2 + 1]$

$z_2(v,s) = 2[(vs)^2 + 4s^2 - 2vs + 1]$

On solving the system of equation (ii), we have $\alpha = \frac{vs}{2} + 2, r = \frac{vs}{2} - 2$

Since our aim is to find integer values, it seen that α and r are integers when either v is even or s is even.

Let v=2V then $\alpha = Vs + 2, r = Vs - 2$, In this case, the corresponding sides of T_1 and T_2 are obtained as

$x_1(V,s) = 16(V^2 + Vs)$

$y_1(V,s) = 8(2Vs + s^2)$

$z_1(V,s) = 4(4V^2 + 2s^2 + 4Vs)$

$x_2(V,s) = 8Vs^2 - 16s$

$y_2(V,s) = 2[(Vs)^2 - 4Vs - 4s^2 + 4]$

$z_2(V,s) = 2[(Vs)^2 + 4s^2 - 4Vs + 4]$

CHOICE: 3

Considering $v = 2^{2k}s$ in (6), it is written as

$\alpha^2 = r^2 + (2^{2k+1}s)^2$ (7)

which is the well known Pythagorean equation which is satisfied by

$$\alpha = x^2 + y^2, r = x^2 - y^2, s = \frac{xy}{2^k}, x > y > 0$$

Replacing x by $2^k x$ in the above equations, we have

$$\alpha = (2^k x)^2 + y^2, r = (2^k x)^2 - y^2, s = xy$$

Thus, in view of (6) and using (1), the corresponding sides of T_1 and T_2 are found to be

$x_1(x,y,k) = 4x^2y^2[(2^{2k} + 1)^2 - 1]$

$y_1(x,y,k) = 8x^2y^2(2^{2k} + 1)$

$z_1(x,y,k) = 4x^2y^2[(2^{2k} + 1)^2 + 1]$

$x_2(x,y,k) = 2^{2k+2}x^3y - 8xy^3$

$y_2(x,y,k) = 2[2^{2k}x^2 - y^2 + 2xy][2^{2k}x^2 - y^2 - 2xy]$

$z_2(x,y,k) = 2[2^{4k}x^4 + y^4 + 4x^2y^2 - 2^{2k+1}x^2y^2]$

Note: 4

The solution of (7) may also be taken as

$$\alpha = x^2 + y^2, r = 2xy, s = \frac{1}{2^k}(x^2 - y^2), k > 0$$

Replacing x by $2^k x$ and y by $2^k y$ in above equations, we get

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$$\alpha = 2^{2k}[x^2 + y^2], r = 2^{2k+1}xy, s = 2^k(x^2 - y^2)$$

Thus, in view of (6) and using (1), the corresponding sides of T_1 and T_2 are

$$x_1(x,y,k) = 2^{2k+2}(2^{4k} + 2^{2k+1})(x^2 - y^2)^2$$

$$y_1(x,y,k) = 2^{2k+2}(2^{2k} + 1)(x^2 - y^2)^2$$

$$z_1(x,y,k) = 2^{2k+2}(2^{4k} + 2^{2k+1} + 2)(x^2 - y^2)^2$$

$$x_2(x,y,k) = 2^{3k+3}xy(x^2 - y^2)$$

$$y_2(x,y,k) = 2^{4k+3}x^2y^2 - 2^{2k+1}(x^2 - y^2)^2$$

$$z_2(x,y,k) = 2[2^{4k+2}x^2y^2 + 2^{2k}(x^2 - y^2)^2]$$

CHOICE: 4

Introducing of the linear transformations

$$m = 4u + v, n = 4u - v, p = 3u + r, q = 3u - r, u \neq v, r \neq u \quad (8)$$

in (2), it leads to

$$(5u)^2 = \alpha^2 + v^2 + r^2$$

which is satisfied by

$$\alpha = a^2 - b^2 - c^2, r = 2ab, v = 2ac, u = \frac{1}{5}(a^2 + b^2 + c^2)$$

Substituting a= 5A, b=5B and c=5C in the above equations, we get

$$\alpha = 25(A^2 - B^2 - C^2), r = 50AB,$$

$$v = 50AC, u = 5(A^2 + B^2 + C^2)$$

In view of (8) and using (1), the corresponding sides of T_1 and T_2 are as follows

$$x_1(A,B) = 400AC(A^2 + B^2 + C^2)$$

$$y_1(A,B) = 2[(20A^2 + 20B^2 + 20C^2)^2 - 2500A^2C^2]$$

$$z_1(A,B) = 2[(20A^2 + 20B^2 + 20C^2)^2 + 2500A^2C^2]$$

$$x_2(A,B) = 150AB(A^2 + B^2 + C^2)$$

$$y_2(A,B) = 2[(15A^2 + 15B^2 + 15C^2)^2 - 2500A^2B^2]$$

$$z_2(A,B) = 2[(15A^2 + 15B^2 + 15C^2)^2 + 2500A^2B^2]$$

CHOICE:5

Note that (2) is satisfied by

$$m = a^2 - b^2, n = a^2 - b^2 - 4ab,$$

$$p = 2a^2 + 2ab, q = 2ab - 2b^2, a > 4b > 0$$

Hence the corresponding sides of T_1 and T_2 are obtained as

$$x_1(a,b) = 8ab(a^2 - b^2) - 16a^2b^2$$

$$y_1(a,b) = 2(a^2 - b^2)^2 - 4ab(a^2 - b^2)$$

$$z_1(a,b) = 2(a^2 + b^2)^2 + 8a^2b^2 - 8ab(a^2 - b^2)$$

$$x_2(a,b) = 4(a^4 - b^4 - 2a^3b + 2ab^3)$$

$$y_2(a,b) = 8ab(a^2 - b^2)$$

$$z_2(a,b) = 4(a^4 + b^4 + 2a^2b^2 + 2a^3b - 2ab^3)$$

IV. CONCLUSION

One may search for pairs of Pythagorean triangles such that the sum of the product of generators is represented by special polygonal numbers and pyramidal numbers.

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