Pythagorean Triangle with Hypotenuse -4(Area/Perimeter) is 3 Times a Square Integer

M.A. Gopalan, S. Vidhyalakshmi, N. Thiruniraiselvi

Abstract:- We present infinity many Pythagorean triangles, where, in each, the hypotenuse -4(Area/Perimeter) is a square multiple of 3. A few numerical examples are presented. Also, a few interesting relations among the sides of the Pythagorean triangles are given. Further, by considering suitable linear combination among the generators of the Pythagorean triangles, Diophantine 3-tuples and special dio-3 tuples with suitable property are obtained.

Index Terms—Area/perimeter, Pythagorean triangle, square integer.

I. INTRODUCTION

The Pythagorean numbers play a significant role in the theory of higher arithmetic as they come in the majority of indeterminate problems; had a marvelous effect on a credulous people and always occupy a remarkable position due to unquestioned historical importance. The method of obtaining three non-zero integers *x*, *y* and *z* under certain relations satisfying the relation $x^2+y^2=z^2$ has been a matter of interest to various Mathematicians [1-6]. In [7-20], special Pythagorean problems are studied. In this communication, we search for patterns of Pythagorean triangles, where, in each, of the hypotenuse -4(Area/Perimeter) is a square multiple of 3.Some numerical examples are presented. A few interesting relations among the sides of the Pythagorean triangles are given. Further, by considering suitable linear combination among the generators of the Pythagorean triangles, Diophantine 3-tuples and special dio-3 tuples with suitable property are obtained.

Notations:

- $t_{m,n}$ - Polygonal number of rank n with side m.
- $cp_{m,n}$ – Centered polygonal number of rank n with side m.
- $f^{r}_{m,n}$ – m-dimensional figurate number

II. METHOD OF ANALYSIS

The most cited solution of the Pythagorean equation,

$$x^2+y^2=z^2 \quad (1)$$

is represented by

$$x=2pq, y=p^2-q^2, z=p^2+q^2 \quad (2)$$

where $p>q>0$ (3)

Denote the area and perimeter by A and P respectively.

The assumption $z-\frac{4A}{P}=3\alpha^2$ (*)

leads to the equation $(p-q)^2+2q^2=3\alpha^2$ (4)

Let $\alpha=a^2+2b^2$ (5)

Write 3 as 3 $3=(1+i\sqrt{2})(1-i\sqrt{2})$ (6)

Using (5) and (6) in (4), and employing the method of factorization, define

$$(p-q)+i\sqrt{2}q=(1-i\sqrt{2})(a+i\sqrt{2}b)^2$$

Equating the real and imaginary parts, we get

$$p=2a^2-4b^2-2ab$$
$$q=a^2-2ab^2+2ab \quad (7)$$

The condition (3) will be satisfied when

$$a^2-2b^2>4ab$$

The above inequality holds good provided

(i) $a=n+9s-5, b=2s-1$ (8)

(ii) $a=n+9s-1, b=2s$ (9)

Substituting the corresponding values of a,b in (7), we obtain the values of p and q.

Thus, in view of (2), the sides of the Pythagorean triangle satisfying (*) are obtained.

The above process is illustrated below:

Illustration -1:

Substituting (8) in (7), we have

$$p=2n^2+110s^2+32ns-126s-18n+36$$

$$q=n^2+119s^2+22ns-120s-12n+33$$

Employing (2), the sides of the Pythagorean triangle satisfying (*) are given by

Manuscript Received on April 2015.

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$$x(n,s)=4n^4+152n^3s+2064n^2s^2+11816ns^3+23980s^4-84n^3-2292n^2s+11040ns-19788ns^2-53868s^3+636n^2+45348s^2-16956s-2052n+2376$$

$$y(n,s)=3n^4+84n^3s+516n^2s^2+2244ns^3+219s^4-48n^3-642n^2s-4128ns^2-1560s^3+258n^2+2202s^2+2508ns-1152s-504n+207$$

$$z(n,s)=5n^4+23981s^4+2166n^2s^2+172n^3s+11836ns^3-96n^3-2424n^2s-19920ns^2-53880s^3+11172ns+45390s^2-678n^2-16992s-2088n+2385$$

A few numerical examples are presented below

Table-1

n	s	p	q	z	$\frac{2A}{P}$	$z-\frac{4A}{P}$
1	1	36	33	2385	99	$3(27)^2$
2	2	324	297	193185	8019	$3(243)^2$
3	2	380	334	255956	15364	$3(274)^2$

Illustration-2:

Substituting (9) in (7), we have

$$p=2n^2+110s^2+32ns-32s-4n+2$$

$$q=n^2+109s^2+22ns-22s-2n+1$$

Using (2), the sides of the Pythagorean triangle satisfying (*) are given by

$$x(n,s)=4n^4+152n^3s+2064n^2s^2+23980s^4+11816ns^3-16n^3-456n^2s-4128ns^2-11816s^3+24n^2+2064s^2+456ns-16n-152s+4$$

$$y(n,s)=3n^4+84n^3s+762n^2s^2+2244ns^3+219s^4-12n^3-252n^2s-1524ns^2-2244s^3+18s^2+762s^2+252ns-84s-12n+3$$

$$z(n,s)=5n^4+23981s^4+172n^3s+2166n^2s^2+11836ns^3-20n^3-516n^2s-4332ns^2-11836s^3+30n^2+2166s^2+516ns-20n-172s+5$$

A few numerical examples are presented below

Table -2

n	s	p	q	z	$\frac{2A}{P}$	$z-\frac{4A}{P}$
1	1	110	109	23981	109	$3(81)^2$
2	1	144	132	38160	1584	$3(108)^2$
2	2	506	481	487397	12025	$3(393)^2$

The sides of the Pythagorean triangles presented in illustration (1) and (2) satisfy the following properties:

(1) Each of the following relations represents a nasty number:

(i) $2(2x-x-y)$

(ii) $2(z-\frac{4A}{p})$

(iii) $3(x-\frac{4A}{P})$

(iv) $6(y-\frac{4A}{P})$

(v) $3(4x+3y+5z)$

(2) Each of the following expressions is written as sum of two squares:

(i) $2x+2y+3z$

(ii) $2x+y+3z$

(3) $4x+3y+7z$ represents sum of 3 squares.

(4) $x+y+z=\frac{4A}{P}$

To analyze the various properties satisfied by illustrations (1 and 2), we have to going for particular values for either n or s. in particular, taking s=1 in (Illustration-1), we have

$$x(n,1) = 24(F^n_{4,3} + F^n_{4,5}) + 52cp_{6,n} + 388t_{4,n}$$
$$\equiv 80(\bmod 100)$$
$$y(n,1) = 24F^n_{4,5} + 26cp_{6,n} + 123t_{4,n}$$
$$\equiv 0(\bmod 2)$$
$$z(n,1) = 24F^n_{4,7} + 62cp_{6,n} + 413t_{4,n} - 884$$
$$\equiv 0(\bmod 501)$$

Similarly, taking s=1in (Illustration-2), we get

$$x(n,1) = 24(F^n_{4,3} + F^n_{4,5}) + 120cp_{6,n}$$
$$\equiv 0(\bmod 2)$$
$$y(n,1) = 24F^n_{4,5} + 62cp_{6,n} + 519t_{4,n} + 1344$$
$$\equiv 0(\bmod 958)$$
$$z(n,1) = 24F^n_{4,7} + 138cp_{6,n} + 1673t_{4,n}$$
$$\equiv 0(\bmod 2)$$

Remarkable observations:

Remark-(Illustration –1)

1. Let

$a = p(n,1) - q(n,1) + 2, b = q(n,1) - 22$
$, c = 2(a+b)$

then (a,b,c) represents Diophantine -3-tuple with property D(9n ^{2})

Proof:

$$ab + 9n^2 = (n^2 + 4n)(n^2 + 10n) + 9n^2$$
$$= n^4 + 14n^3 + 49n^2$$
$$= (n^2 + 7n)^2$$
$$ac + 9n^2 = (n^2 + 4n)(4n^2 + 28n) + 9n^2$$
$$= 4n^4 + 44n^3 + 121n^2$$
$$= (2n^2 + 11n)^2$$
$$bc + 9n^2 = (n^2 + 10n)(4n^2 + 28n) + 9n^2$$
$$= 4n^4 + 68n^3 + 289n^2$$
$$= (2n^2 + 17n)^2$$

Thus, the triple (a,b,c) is a Diophantine-3-tuple with property D($9n^2$), because the product of any two members of the above set added with ($9n^2$) is a perfect square.

2.Let

$a = p(n,1) - n^2 - 20, b = q(n,1) - 22, c = 2(a+b)$

then (a,b,c) represents Diophantine-3-tuple with property D(4n ^{2})

Proof:

$$ab + 4n^2 = (n^2 + 14n)(n^2 + 10n) + 4n^2$$
$$= n^4 + 24n^3 + 144n^2$$
$$= (n^2 + 12n)^2$$
$$ac + 4n^2 = (n^2 + 14n)(4n^2 + 48n) + 4n^2$$
$$= 4n^4 + 104n^3 + 676n^2$$
$$= (2n^2 + 26n)^2$$
$$bc + 4n^2 = (n^2 + 10n)(4n^2 + 48n) + 4n^2$$
$$= 4n^4 + 88n^3 + 484n^2$$
$$= (2n^2 + 22n)^2$$

3.Let

$a = p(n,1) - 20, b = 2q(n,1) - 44, c = 2(a+b)$

then (a,b,c) represents Diophantine-3-tuple with property D(9n ^{2})

Proof:

$$ab + 9n^2 = (2n^2 + 14n)(2n^2 + 20n) + 9n^2$$
$$= 4n^4 + 289n^3 + 68n^2$$
$$= (2n^2 + 17n)^2$$
$$ac + 9n^2 = (2n^2 + 14n)(8n^2 + 68n) + 9n^2$$
$$= 16n^4 + 248n^3 + 961n^2$$
$$= (4n^2 + 31n)^2$$
$$bc + 9n^2 = (2n^2 + 20n)(8n^2 + 68n) + 9n^2$$
$$= 16n^4 + 296n^3 + 1369n^2$$
$$= (4n^2 + 37n)^2$$

4.Let

$a = p(n,1) - q(n,1) + 2, b = q(n,1) - 22$
$, c = 2(a+b) + 3$

then (a,b,c) represents **special Dio-3-tuple** with property D(9n ^{2} +1)

Proof:

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$$ab+(a+b)+(9n^2+1)=(n^2+4n)$$
$$*(n^2+10n)+(2n^2+14n)+(9n^2+1)$$
$$=n^4+14n^3+51n^2+14n+1$$
$$=(n^2+7n+1)^2$$

$$ac+(a+c)+(9n^2+1)=(n^2+4n)$$
$$*(4n^2+28n+3)+(5n^2+32n+3)+(9n^2+1)$$
$$=4n^4+44n^3+129n^2+44n+4$$
$$=(2n^2+11n+2)^2$$

$$bc+(b+c)+(9n^2+1)=(n^2+10n)$$
$$*(4n^2+28n+3)+(5n^2+38n+3)+(9n^2+1)$$
$$=4n^4+68n^3+297n^2+68n+4$$
$$=(2n^2+17n+2)^2$$

Thus, the triple (a,b,c) is a special dio-3-tuple with property D($9n^2$+1), because the product of any two members of the above set added with the same members and increased by ($9n^2$+1) is a perfect square.

Remark-(Illustration –2)

1. Let $a=p(n,1)-n^2-80, b=q(n,1)-88$ $, c=2(a+b)$

   then (a,b,c) represents Diophantine -3-tuple with property D($16n^2$)

Proof:

$$ab+16n^2=(n^2+28n)(n^2+20n)+16n^2$$
$$=(n^2+24n)^2$$
$$ac+16n^2=(n^2+28n)(4n^2+96n)+16n^2$$
$$=4n^4+208n^3+2704n^2$$
$$=(2n^2+52n)^2$$
$$bc+16n^2=(n^2+20n)(4n^2+96n)+16n^2$$
$$=4n^4+176n^3+1936n^2$$
$$=(2n^2+44n)^2$$

2. Let

   $a=p(n,1)-q(n,1)+8, b=q(n,1)-88$ $, c=2(a+b)$

   then (a,b,c) represents Diophantine -3-tuple with property D($36n^2$)

Proof:

$$ab+36n^2=(n^2+8n)(n^2+20n)+36n^2$$
$$=n^4+28n^3+196n^2$$
$$=(n^2+14n)^2$$
$$ac+36n^2=(n^2+8n)(4n^2+56n)+36n^2$$
$$=4n^4+88n^3+484n^2$$
$$=(2n^2+22n)^2$$
$$bc+36n^2=(n^2+20n)(4n^2+56n)+36n^2$$
$$=4n^4+136n^3+1156n^2$$
$$=(2n^2+34n)^2$$

3. Let $a=p(n,1)-80, b=2q(n,1)-176$ $, c=2(a+b)$

   then (a,b,c) represents Diophantine -3-tuple with property D($36n^2$)

Proof:

$$ab+36n^2=(2n^2+28n)(2n^2+40n)+36n^2$$
$$=4n^4+136n^3+1156n^2$$
$$=(2n^2+34n)^2$$
$$ac+36n^2=(2n^2+28n)(8n^2+136n)+36n^2$$
$$=16n^4+496n^3+3844n^2$$
$$=(4n^2+62n)^2$$
$$bc+36n^2=(2n^2+40n)(8n^2+136n)+36n^2$$
$$=16n^4+592n^3+5476n^2$$
$$=(4n^2+74n)^2$$

4. Let $a=p(n,1)-q(n,1)+8, b=q(n,1)-88$ $, c=2(a+b)-33$

   then (a,b,c) represents **special Dio-3-tuple** with property D(289-504n)

Proof:

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$$ab+(a+b)+(289-504n)=(n^2+8n)$$
$$*(n^2+20n)+(2n^2+28n)+(289-504n)$$
$$=n^4+28n^3+162n^2-476n+289$$
$$=(n^2+14n-17)^2$$

$$ac+(a+c)+(289-504n)=(n^2+8n)$$
$$*(4n^2+56n-33)+(5n^2+64n-33)$$
$$+(289-504n)$$
$$=4n^4+88n^3+420n^2-704n+256$$
$$=(2n^2+22n-16)^2$$

$$bc+(b+c)+(289-504n)=(n^2+20n)$$
$$*(4n^2+56n+33)+(5n^2+76n+33)$$
$$+(289-504n)$$
$$=4n^4+136n^3+1092n^2-1088n+256$$
$$=(2n^2+34n-16)^2$$

III. CONCLUSION

In this paper, we obtained infinitely many Pythagorean triangles, where, in each, the hypotenuse -4(Area/Perimeter) is a square multiple of 3.Also, from the generators of the Pythagorean triangles, dio-3-tuples and special dio-3-tuples with suitable property are obtained. In conclusion, one may search for Pythagorean triangles with the combination among the sides, area and perimeter satisfy special numbers, namely, polygonal numbers and pyramidal numbers.

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