On the Non-Homogeneous Quintic Equation with Seven Unknowns $xy(x^2+y^2)+zw(z^2+w^2)=(X^2+Y^2)T^3$

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Abstract— ***We obtain infinitely many non-zero integer solutions (x,y,z,w,X,Y,T) satisfying the non- homogeneous quintic equation with seven unknowns given by $xy(x^2+y^2)+zw(z^2+w^2)=(X^2+Y^2)T^3$. Various interesting relations between the solutions and special numbers, namely, polygonal numbers, Pyramidal numbers, Stella Octangular numbers, Octahedral numbers,, Jacobsthal number, Jacobsthal-Lucas number, keynea number, Centered pyramidal numbers are presented***

Index Terms— ***Centered pyramidal numbers, Integral solutions, Non-homogeneous Quintic equation, Polygonal numbers, Pyramidal numbers***

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Notations:

$T_{m,n}$ - Polygonal number of rank n with size m

SO_n - Stella Octangular number of rank n

J_n - Jacobsthal number of rank of n

KY_n - Keynea number of rank n

$CP_{n,6}$ - Centered hexagonal pyramidal number of rank n

P_n^m - Pyramidal number of rank n with size m

OH_n - Octahedral number of rank n

j_n - Jacobsthal-Lucas number of rank n

I. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-3]. For illustration, one may refer [4-10] for homogeneous and non-homogeneous quintic equations with three, four and five unknowns. This paper concerns with the problem of determining non-trivial integral solution of the non-homogeneous quintic equation with seven unknowns given by $xy(x^2+y^2)+zw(z^2+w^2)=(X^2+Y^2)T^3$. A few relations between the solutions and the special numbers are presented

Initially, the following two sets of solutions in (x,y,z,w,X,Y,T) satisfy the given equation:

$$(2k^2+k+p,\ 2k^2+k-p,\ p-2k^2+k,\ p+2k^2-k,\ 2k,\ 4k^2,\ 2k),$$

$$(2k^2+3k+p+1,\ 2k^2+3k-p+1,\ p-2k^2-k,\ p+2k^2+k,\ 2k+1,\ 4k^2+4k+1,\ 2k+1)$$

However we have other patterns of solutions, which are illustrated below:

II. METHOD OF ANALYSIS

The Diophantine equation representing the non-homogeneous quintic equation is given by

$$xy(x^2+y^2)+zw(z^2+w^2)=(X^2+Y^2)T^3 \quad (1)$$

Introduction of the transformations

$$x=u+p,\ y=u-p,\ z=p+v,\ w=p-v,\ X=u+v,\ Y=u-v \quad (2)$$

in (1) leads to

$$u^2-v^2=T^3 \quad (3)$$

The above equation (3) is solved through different approaches and thus, one obtains different sets of solutions to (1)

A. *Approach1*

The solution to (3) is obtained as

$$\left.\begin{aligned} u&=a(a^2-b^2)\\ v&=b(a^2-b^2)\\ T&=(a^2-b^2)\end{aligned}\right\} \quad (4)$$

In view of (2) and (4), the corresponding values of x,y,z,w,X,Y,T are represented by

$$\left.\begin{aligned} x&=a(a^2-b^2)+p\\ y&=a(a^2-b^2)-p\\ z&=p+b(a^2-b^2)\\ w&=p-b(a^2-b^2)\\ X&=(a^2-b^2)(a+b)\\ Y&=(a^2-b^2)(a-b)\\ T&=a^2-b^2\end{aligned}\right\} \quad (5)$$

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The above values of x, y, z, w, X, Y and T satisfies the following relations:

1. $[x(a,b)+y(a,b)][X(a,b)+Y(a,b)]$ is a perfect square.
2. The following expressions are nasty numbers:

(a) $3[X(a(a+1),a)+Y(a(a+1),a)-4T_{3,a^2}-6CP_{a,6}]$

(b) $3[x(a(a+1),a)+y(a(a+1),a)+z(a(a+1),a)-w(a(a+1),a)]$

(c) $2[x(2a,a)+y(2a,a)-6SO_a+12T_{3,a}]$.

3. The following expressions are cubic integers

(a) $9[x(2a,a)+y(2a,a)+z(2a,a)+w(2a,a)+X(2a,a)+Y(2a,a)]$

(b) $4[Y(a,1)-X(a,1)+2T_{4,a}]$

4. $w(a,1)-z(a,1)+T(a,1)+T_{4,a}=1$
5. $x(2a,a)+y(2a,a)+T(2a,a)=12CP_{a,6}+6T_{3,a}-3SO_a$
6. $T(2a,a)+z(2a,a)+w(2a,a)-3T_{4,a}=0(\bmod 2)$
7. $T(2^{2n+1},2^{2n})+3J_{2n+1}=3KY_{2n}$

B. Approach2

The assumptions

$$u=UT, v=VT \quad (6)$$

in (3) yields to

$$U^2-V^2=T \quad (7)$$

(i) Taking $T=-t^2$ (8)
in (7), we get

$$U^2+t^2=V^2 \quad (9)$$

Then the solution to (9) is given by

$$t=2\alpha\beta, V=\alpha^2+\beta^2, U=\alpha^2-\beta^2, \alpha>\beta>0 \quad (10)$$

$$U=2\alpha\beta, V=\alpha^2+\beta^2, t=\alpha^2-\beta^2, \alpha>\beta>0 \quad (11)$$

From (6), (8) and (10) we get,

$$\left.\begin{aligned} u&=-4\alpha^2\beta^2(\alpha^2-\beta^2)\\ v&=-4\alpha^2\beta^2(\alpha^2+\beta^2)\\ T&=-4\alpha^2\beta^2 \end{aligned}\right\} \quad (12)$$

In view of (12) and (2), we get the corresponding integral solution of (1).as

$$\left.\begin{aligned} x&=-4\alpha^2\beta^2(\alpha^2-\beta^2)+p\\ y&=-4\alpha^2\beta^2(\alpha^2-\beta^2)-p\\ z&=p-4\alpha^2\beta^2(\alpha^2+\beta^2)\\ w&=p+4\alpha^2\beta^2(\alpha^2+\beta^2)\\ X&=-8\alpha^4\beta^2\\ Y&=8\alpha^2\beta^4\\ T&=-4\alpha^2\beta^2 \end{aligned}\right\} \quad (13)$$

Properties:

1. $x(2a,a)+y(2a,a)+96(2P_a^5-2T_{3,a^2}+T_{4,a})=0$
2. $X(2^{2n},2^{2n})+Y(2^{2n},2^{2n})+z(2^{2n},2^{2n})-w(2^{2n},2^{2n})+T(2^{2n},2^{2n})+48J_{6n}+4J_{4n}+12=0$
3. $3\alpha[4T_{4,\alpha^2}+x(\alpha,\alpha)-y(\alpha,\alpha)+T(\alpha,\alpha)]$ is a nasty number
4. $2\alpha^2[x(\alpha,\beta)-y(\alpha,\beta)+z(\alpha,\beta)+w(\alpha,\beta)]$ is a Cubic number
5. $2T(\alpha,\alpha)[y(\alpha,\alpha)-x(\alpha,\alpha)-X(\alpha,\alpha)-Y(\alpha,\alpha)-z(\alpha,\alpha)-w(\alpha,\alpha)]$ is a quintic integer

Remark1:

Similarly by considering (6), (8), (11) and (2), we get the corresponding integral solution to (1) as

$$\begin{aligned} x&=2\alpha\beta T+p\\ y&=2\alpha\beta T-p\\ z&=p+(\alpha^2+\beta^2)T\\ w&=p-(\alpha^2+\beta^2)T\\ X&=(\alpha+\beta)^2T\\ Y&=-(\alpha-\beta)^2T\\ T&=-(\alpha^2-\beta^2)^2 \end{aligned}$$

ii) Now, rewrite (9) as,

$$U^2+t^2=1*V^2 \quad (14)$$

Also 1 can be written as

$$1=(-i)^n(i)^n \quad (15)$$

Let $V=a^2+b^2$ (16)

Substituting (15) and (16) in (14) and using the method of factorization, define,

$$(U+it)=i^n(a+ib)^2 \quad (17)$$

Equating real and imaginary parts in (17) we get

$$\left.\begin{aligned} U &= \cos\frac{n\pi}{2}(a^2-b^2) - 2ab\sin\frac{n\pi}{2} \\ t &= 2ab\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}(a^2-b^2) \end{aligned}\right\} \quad (18)$$

In view of (2), (6), (8) and (18), the corresponding values of x, y, z, w, X, Y, T are represented as

$$\left.\begin{aligned} x &= T[\cos\frac{n\pi}{2}(a^2-b^2) - 2ab\sin\frac{n\pi}{2}] + p \\ y &= T[\cos\frac{n\pi}{2}(a^2-b^2) - 2ab\sin\frac{n\pi}{2}] - p \\ z &= p + a^2 + b^2 \\ w &= p - a^2 - b^2 \\ X &= T[\cos\frac{n\pi}{2}(a^2-b^2) - 2ab\sin\frac{n\pi}{2} + a^2 + b^2] \\ Y &= T[\cos\frac{n\pi}{2}(a^2-b^2) - 2ab\sin\frac{n\pi}{2} - a^2 - b^2] \\ T &= -(\sin\frac{n\pi}{2}(a^2-b^2) + 2ab\cos\frac{n\pi}{2})^2 \end{aligned}\right\} \quad (19)$$

(iii) 1 can also be written as

$$1 = \frac{((m^2-n^2)+i2mn)((m^2-n^2)-i2mn)}{(m^2+n^2)^2} \quad (20)$$

Substituting (16) and (20) in (14) and using the method of factorization, define,

$$(U+it) = \frac{(m^2-n^2)+i2mn)}{(m^2+n^2)^2}(a+ib)^2 \quad (21)$$

Equating real and imaginary parts in (21) we get

$$\left.\begin{aligned} U &= \frac{1}{m^2+n^2}\{(m^2-n^2)(a^2-b^2) - 4mnab\} \\ t &= \frac{1}{m^2+n^2}\{2ab(m^2-n^2) + 2mn(a^2-b^2)\} \end{aligned}\right\} \quad (22)$$

In view of (2), (6), (8) and (22), the corresponding values of x, y, z, w, X, Y, T are represented as

$$\left.\begin{aligned} x &= (m^2+n^2)T[(A^2-B^2)(m^2-n^2) - 4mnAB] + p \\ y &= (m^2+n^2)T[(A^2-B^2)(m^2-n^2) - 4mnAB] - p \\ z &= p + (m^2+n^2)^2(A^2+B^2)T \\ w &= p - (m^2+n^2)^2(A^2-B^2)T \\ X &= (m^2+n^2)T[(m^2-n^2)(A^2-B^2) - 4mnAB + (m^2+n^2)(A^2+B^2)] \\ Y &= (m^2+n^2)T[(m^2-n^2)(A^2-B^2) - 4mnAB - (m^2+n^2)(A^2+B^2)] \\ T &= -(m^2+n^2)^2[2AB(m^2-n^2) + 2mn(A^2-B^2)]^2 \end{aligned}\right\} \quad (23)$$

C. *Approach3*

Equation (3) can be written as

$$(u-v)(u+v) = 1 \times T^3$$

(24)

Writing (24) as a set of double equations in two different ways as shown below, we get

Set1: $u+v=T^3,\ u-v=1$

Set2: $u-v=T^3,\ u+v=1$

Solving **set1,** the corresponding values of u, v *and* T are given by

$$u = 4k^3+6k^2+3k+1, v = 4k^3+6k^2+3k, \quad T = 2k+1 \quad (25)$$

In view of (25) and (2), the corresponding solutions to (1) obtained from set1 are represented as shown below:

$$x = 4k^3+6k^2+3k+1+p$$
$$y = 4k^3+6k^2+3k+1-p$$
$$z = p+4k^3+6k^2+3k$$
$$w = p-4k^3-6k^2-3k$$
$$X = 8k^3+12k^2+6k+1$$
$$Y = 1$$
$$T = 2k+1$$

Properties:

1. $x(a)+y(a)-24P_a^4 = 0 \pmod 2$
2. $2[x(a)+y(a)+z(a)+w(a)+T(a)- 24P_a^4 - 6T_{4,a} + 2T_{8,a} - 2\alpha]$ is a nasty number.
3. $4[X(a)+Y(a)-8P_a^8-16T_{3,a}-6(OH)_a+4CP_{a,6}]$ is a cubic integer.
4. $X(a)-Y(a)+x(a)-y(a)+z(a)+w(a)- 24P_a^4-4CP_{a,6}+2SO_a \equiv 0 \pmod 4)$
5. $4[x(a)+y(a)+Y(a)+T(a)-48P_a^3+ 24T_{3,a}-12(OH_a)+8CP_{a,6}]$ is a biquadratic integer

Remark2:

Similarly, the solutions corresponding to set2 can also be obtained.

D. *.Approach4*

Substituting $T = a^2-b^2$ in (3) and writing it as a system of double equations as

$$u+v = (a+b)^3$$
$$u-v = (a-b)^3$$

and solving, we get

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$$\left.\begin{aligned} u &= a^3+3ab^2 \\ v &= 3a^2b+b^3 \end{aligned}\right\} \quad (26)$$

In view of (26) and (2), the corresponding solutions to (1) are represented as shown below:

$x = a^3+3ab^2+p$

$y = a^3+3ab^2-p$

$z = p+3a^2b+b^3$

$w = p-3a^2b-b^3$

$X = a^3+3ab^2+3a^2b+b^3$

$Y = a^3+3ab^2-3a^2b-b^3$

$T = a^2-b^2$

Properties:

1. $4[x(a,a)+y(a,a)+z(a,a)-w(a,a)]$ is a cubic integer
2. $X(a,1)+Y(a,1)-4CP_{a,3}-14T_{4,a}+4T_{9,a}=0$
3. $x(a,a)+y(a,a)+z(a,a)-w(a,a)+X(a,a)+Y(a,a)-12SO_a-36T_{4,a}+24T_{5,a}=0$
4. $16j_{6n}-T(2^{2n},2^{2n})-X(2^{2n},2^{2n})-Y(2^{2n},2^{2n})-x(2^{2n},2^{2n})-y(2^{2n},2^{2n})$ is a biquadratic integer
5. $X(2a,a)+Y(2a,a)-2x(2a,a)+2y(2a,a)-2T(2a,a)\equiv 0 \pmod{20}$

III. CONCLUSION

In conclusion, one may search for different patterns of solutions to (1) and their corresponding properties.

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