On the Ternary Cubic Equation $3(x^2+y^2)-2xy+4(x+y)+4=51z^3$

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Abstract—The non-homogeneous ternary cubic Diophantine equation given by $3(x^2+y^2)-2xy+4(x+y)+4=51z^3$ is considered. Different patterns of non-zero distinct integer solutions to the above equation are obtained. For each of these patterns, a few interesting relations between the solutions and the special figurate numbers are obtained.

Keywords: Non-homogeneous, ternary cubic Diophantine equation, integer solutions, polygonal numbers, pyramidal numbers. 2010 Mathematics Subject Classification: 11 D 25

I. INTRODUCTION

Integral solutions for the homogeneous or non – homogeneous Diophantine equation is an interesting concept as it can be seen from [1-3].In particular, one may refer [4-23] for cubic Diophantine equation with three unknowns. This communication concerns with yet another interesting cubic equation with three unknowns $3(x^2+y^2)-2xy+4(x+y)+4=51z^3$ for determining its non-zero distinct integer solutions. A few interesting relations among the solutions are presented.

NOTATIONS

$t_{m,n}$ - Polygonal number of rank n with sides m

$cp_{m,n}$- Centered Polygonal number of rank n with sides m

P_n^m- Pyramidal number of rank n with sides m

CP_n^m- Centered Pyramidal number of rank n with sides m

PCS_n^m - Prism number of rank n with sides m

$P(n)$- Pronic number of rank n

$G(n)$- Gnomonic number

$CO(n)$- Centered Octahedral number

$I(n)$- Icosahedral number

$D(n)$- Dodecahedral number

$SO(n)$- Stella octangula number

$RD(n)$- Rhombic Dodecahedral number

$CD(n)$- Centered Dodecahedral number

$CC(n)$- Centered Cube number

$TTH(n)$- Truncated Tetrahedral number

$TOH(n)$-Truncated Octahedral number

$g_m(n)$- m-gram number of rank n

II. METHOD OF ANALYSIS

The ternary cubic equation to be solved is

$$3(x^2+y^2)-2xy+4(x+y)+4=51z^3 \qquad (1)$$

Introducing the linear transformations

$$x=u+v \quad ; \quad y=u-v \quad (u\neq v\neq 0) \qquad (2)$$

in (1), it is written as

$$(2u+2)^2+8v^2=51z^3 \qquad (3)$$

Now (3) is solved through different ways and using (2), different patterns of integer solutions to (1) are obtained.

PATTERN I

Assume $z=a^2+8b^2$ (4)

Write 51 as $51=(7+i\sqrt{2})(7-i\sqrt{2})$ (5)

Using (4) and (5) in (3) and applying the method of factorization, define

$$(2u+2)u+i2\sqrt{2}v=(7+i\sqrt{2})\left(a+i2\sqrt{2}b\right)^3$$

Equating the real and imaginary parts, the values of *u* and *v* are obtained as

$$u=\frac{1}{2}[7a^3-12a^2b-168ab^2+32b^3-2]$$
$$v=\frac{1}{2}[a^3+42a^2b-24ab^2-112b^3]$$

Substituting the above values of *u* and *v* in (2), the values of *x* and *y* are given by

$$\left.\begin{aligned} x&=4a^3+15a^2b-96ab^2-40b^3-1\\ y&=3a^3-27a^2b-72ab^2+72b^3-1\end{aligned}\right\} \qquad (6)$$

Thus (6) and (4) represent the non-zero distinct integer solutions to (1).

Properties

- $x(a,1)+15z(a,1)-2SO(a)-t_{62,a}\equiv 14\ (mod\ 65)$
- $x(1,a+1)+y(1,a+1)-16SO(a)+144t_{3,a}\equiv -143(mod\ 164)$
- $x(a,1)+z(a,1)-2SO(a)-8t_{6,a}\equiv -33\ (mod\ 86)$
- $x(1,b)-y(1,b)+z(1,b)+6CP_b^{30}+6CP_b^{24}+58CP_b^6$ is a Kynea prime.
- $y(1,b)+9z(1,b)-27SO(b)-18CP_a^6$ is a perfect square.

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PATTERN II

Write (3) as $(2u+2)^2+8v^2=51z^3*1$ (7)

Write 1 as

$$1=\frac{(1+i2\sqrt{2})(1-i2\sqrt{2})}{9} \quad (8)$$

Using (5) and (8) in (7) and repeating the procedure as in pattern I, the corresponding integer solutions (1) are given by

$$x=3a^3-27a^2b-72ab^2+72b^3-1$$
$$y=-2a^3-33a^2b+48ab^2+88b^3-1$$
$$z=a^2+8b^2$$

Properties

- $y(1,b)-x(1,b)-z(1,b)-60t_{3,b-1}-115t_{4,b}+1$ is a Truncated Octahedral number of rank a.
- $x(1,b)+y(1,b)-2PCS_b^{30}-6P_b^{30}-6t_{3,b-1}\equiv -1(mod\ 19)$.
- $y(a,a+1)-10CD(a)-CP_a^6-177P(a)\equiv 12(mod\ 77)$.
- $z(a(a+1),a^2+a+1)-8CP_{9,a(a+1)}-28P(a)$ is a Nasty number.
- $z(a(a+1),a^2+a+1)-x(a(a+1),a^2+a+$ 1−6RDaa+1+152Pa=−57

PATTERN III

Write 1 as

$$1=\frac{(7+i6\sqrt{2})(7-i6\sqrt{2})}{121} \quad (9)$$

Using (5) and (9) in (7) and following the procedure similar to pattern II, the corresponding integer solutions to (1) are obtained as

$$x=5203A^3-22143A^2B-124872AB^2+59048B^3-1$$
$$y=-726A^3-49005A^2B+17424AB^2+130680B^3-1$$
$$z=121A^2+968B^3$$

Properties

- $z(A,1)-y(A,1)-396P_A^{13}-g_{17416}(A)-31707t_{4,A}=-129712$.
- $x(A(A+1),1)-z(A(A+1),1)-1836CP_{A(A+1)}^{17}+37312t_{3,A(A+1)}+106216P(A)-58079$is a Centered hexagonal pyramidal number of rank$A(A+1)$.
- $x(A(A+1),1)+y(A(A+1),1)-2442CP_{A(A+1)}^{11}+142296t_{3,A(A+1)}+34265P(A)=18972$.
- $y(1,B)-x(1,B)-26862CP_B^{16}-17908P(B-$ 1−124388t4,B=−5929.
- $x(1,B)-15396TTH(B)-15SO(B)+47803P(B)+7772t_{4,B}$ is three times a nasty number.

PATTERN IV

1 can also be written as

$$1=\frac{(1+i12\sqrt{2})(1-i12\sqrt{2})}{289} \quad (10)$$

Using (5) and (10) in (7) and repeating the process as above, we obtain the distinct non-zero solutions to (1) as

$$x=16A^3-264A^2B-384AB^2+704B^3-1$$
$$y=-24A^3-216A^2B+576AB^2+576B^3-1$$
$$z=4A^2+32B^2$$

Properties

- $x(A,1)-y(A,1)-60P_A^6+78P(A)\equiv 128\ (mod\ 872)$.
- $x(A,1)+z(A,1)-32P_A^5+276P(A)\equiv 87\ (mod\ 108)$.
- $y(1,B)-1152P_B^5\equiv -25\ (mod\ 216)$.

PATTERN V

It is noted that 51 is also represented by

$$51=(1+i5\sqrt{2})(1-i5\sqrt{2}) \quad (11)$$

Using (8) and (11) in (7) and following the procedure as in pattern II, we obtain the integer solutions to (1) as

$$x=-54A^3-891A^2B+1296AB^2+2376B^3-1$$
$$y=-117A^3+135A^2B+2808AB^2-360B^3-1$$
$$z=9A^2+72B^2$$

Properties

- $x(A,1)-y(A,1)-31SO(A)-CP_A^6+2052t_{3,A}\equiv 6\ (mod\ 455)$.
- $y(A,1)-z(A+1,A)+180SO(A)-126t_{3,A-1}+2664t_{4,A}=-127$.
- $x(A,1)-2z(A+1,A)+108P_A^5+1998t_{3,A-1}\equiv 8(mod\ 261)$.
- $x\big(1,B(B+1)\big)+y\big(1,B(B+1)\big)+z\big(1,B(B+$ 1−30CPBB+130−2CPBB+13 $+g_7\big(B(B+1)\big)-77P(B)+1$ is a Dodecahedral number of rank $B(B+1)$.
- $z(A,A+1)-72G(A+1)$ is a perfect square.

PATTERN VI

Consider (9) and (11) in (7).The repetition of process as in pattern III, the integer solutions to (1) are given by

$$x=-5808A^3-392040A^2B+139392AB^2+1045440B^3-1$$
$$y=-45496A^3-84216A^2B+1091904AB^2+224576B^3-1$$
$$z=484A^2+3872B^2$$

Properties

- $z(A,1)-3872$is a perfect square.
- $x(A,1)+11616P_A^5+139392P(A-1)+246840t_{4,A}=1045439$.
- $y(A(A+1),1)+34122P_{A(A+1)}^{10}+134310t_{3,A(A+1)-1}-996314P(A)=224575$.

PATTERN VII

Considering (10) and (11) in (7) and repeating the process in pattern IV, the corresponding non-zero distinct integer solution to (1) are obtained as

$$x=-24A^3-216A^2B+576AB^2+576B^3-1$$
$$y=-32A^3+120A^2B+768AB^2-320B^3-1$$
$$z=4A^2+32B^2$$

Properties

- $x(A,1)-y(A,1)-6CP_A^8+380t_{3,A-1}+146t_{4,A}=496$.
- $z(A,1)-32$ is a perfect square.
- $x(1,B)+y(1,B)-96P_B^{18}-112P(B)-1184t_{4,B}=-58$.

PATTERN VIII

Replacing z by $2w$ in (3), it is written as

$$(u+1)^2+2v^2=102w^3$$

Writing 102 as $102=(10+i\sqrt{2})(10-i\sqrt{2})$ and performing the analysis as given above, the corresponding integer solutions to (1) are found to be

$$x=11a^3+24a^2b-66ab^2-16b^3-1$$
$$y=9a^3-36a^2b-54ab^2+24b^3-1$$
$$z=2a^2+4b^2$$

Properties

- $x(a(a+1),1)+y(a(a+1),1)-6CP^{26}_{a(a+1)}+2CO\big(a(a+1)\big)+CC\big(a(a+1)\big)+24t_{3,a(a+1)}+73P(a)$ is a nasty number.
- $x(a(a+1),1)-w(a(a+1),1)-22P^{5}_{a(a+1)}-22t_{3,a(a+1)-1}+55P(a)=-13.$
- $y(1,b)-x(1,b)-24P^{12}_{b}\equiv 0\ (mod\ 2).$
- $y(1,b)+w(1,b)-2PCS^{24}_{b}+52t_{3,b}\equiv 0(mod\ 2).$

III. CONCLUSION

It is worth mentioning that 1 and 51 can also be written in the following ways:

(i) $1=\frac{(-1+i2\sqrt{2})(-1-i2\sqrt{2})}{9}$

(ii) $1=\frac{(-7+i6\sqrt{2})(-7-i6\sqrt{2})}{121}$

(iii) $1=\frac{(-1+i12\sqrt{2})(-1-i12\sqrt{2})}{289}$

(iv) $51=(-7+i\sqrt{2})(-7-i\sqrt{2})$

(v) $51=(-1+i5\sqrt{2})(-1-i5\sqrt{2}).$

By considering the combinations for 51 and 1, nine more patterns of integer solutions to (1) are obtained. As Diophantine equations are rich in variety, one may search for other choices of equations and their corresponding integer solutions with suitable properties.

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