On the Homogeneous Biquadratic Equation with 5 Unknowns $(x^2-y^2)\left((4k-1)(x^2+y^2)-(4k-2)xy\right)=2(4k-1)(p^2-q^2)z^2$

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Abstract: ***The homogeneous biquadratic equation with five unknowns given by*** $(x^2-y^2)\left((4k-1)(x^2+y^2)-(4k-2)xy\right)=2(4k-1)(p^2-q^2)z^2$ ***is considered and analyzed for finding its non zero distinct integral solutions. Introducing the linear transformations*** $x=u+v,\quad y=u-v,\quad p=2u+v,\quad q=2u-v$ ***and employing the method of factorization different patterns of non zero distinct integer solutions of the equation under the above equation are obtained. A few interesting relations between the integral solutions and the special numbers namely Polygonal numbers, Pyramidal numbers, Centered Polygonal numbers, Centered Pyramidal numbers, Thabit-ibn-Kurrah number, Carol number, Mersenne number are exhibited.***

Keywords: Homogeneous equation, Integral solutions, Polygonal numbers, Pyramidal numbers and Special numbers. 2010 Mathematics Subject Classification Code: 11D25

Notations:

$t_{m,n}$ ***= Polygonal number of rank n with sides m***

p_m^n ***= Pyramidal number of rank n with sides m***

$ct_{m,n}$ ***= Centered Polygonal number of rank n with sides m***

cp_m^n ***= Centered Pyramidal number of rank n with sides m***

g_n ***= Gnomonic number***

Tha_n ***= Thabit-ibn-Kurrah number***

$carl_n$ ***= Carol number***

cul_n ***= Cullen number***

mer_n ***= Mersenne number***

wo_n ***= Woodhall number***

p_n ***= Pronic number***

I. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-3].

In this context one may refer [4-12] for various problems on the biquadratic Diophantine equations. However, often we come across non-homogeneous biquadratic equations and as such one may require its integral solution in its most general form. This paper concern with the homogeneous biquadratic equation with five unknowns $(x^2-y^2)\left((4k-1)(x^2+y^2)-(4k-2)xy\right)=2(4k-1)(p^2-q^2)z^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The biquadratic equation with five unknowns to be solved for its non-zero distinct integral solution is

$$(x^2-y^2)\left((4k-1)(x^2+y^2)-(4k-2)xy\right)=2(4k-1)(p^2-q^2)z^2 \qquad (1)$$

Consider the transformations

$$\left.\begin{aligned} x&=u+v\\ y&=u-v\\ p&=2u+v\\ q&=2u-v \end{aligned}\right\} \qquad (2)$$

On substituting (2) in (1), we get

$$ku^2+v^2(3k-1)=(4k-1)z^2 \qquad (3)$$

Pattern 1:

Assume $z=ka^2+(3k-1)b^2$ (4)

Write (4k-1) as

$$4k-1=\left(\sqrt{k}+i\sqrt{3k-1}\right)\left(\sqrt{k}+i\sqrt{3k-1}\right) \qquad (5)$$

Substituting (4) and (5) in (3) and employing the method of factorization,

Revised Version Manuscript Received on July 07, 2015.

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$$\left(\sqrt{k}u+i\sqrt{3k-1}v\right)\left(\sqrt{k}u-i\sqrt{3k-1}v\right)=\left(\sqrt{k}+i\sqrt{3k-1}\right)\left(\sqrt{k}-i\sqrt{3k-1}\right)\left(ka^2+(3k-1)b^2\right)^2$$

Equating the real and imaginary parts, we get

$$u=ka^2-(3k-1)b^2-2(3k-1)ab \qquad v=ka^2+2ab-(3k-1)b^2$$

Substituting u and v in (2) the values of x, y, p and q are given by

$$\left.\begin{aligned}
x&=x(a,b,k)=2ka^2-2b^2(3k-1)-2(3k-1)ab+2ab\\
y&=y(a,b,k)=-6kab\\
p&=p(a,b,k)=3ka^2-3b^2(3k-1)-12kab+6ab\\
q&=q(a,b,k)=3ka^2-3b^2(3k-1)-12kab
\end{aligned}\right\} \qquad (6)$$

Thus (4) and (6) represent the non-zero distinct integer solutions of (1)

Properties:

1) $x\left(1,1,2^n+1\right)+10mer_n=$ Nasty number
2) $p(a+1,1,k)+9=t_{6,a}+2t_{3,a}$
3) $q(1,2b,1)+t_{18,b}\equiv 3(\mathrm{mod}\,31)$
4) $z(n+1,n,1)-y\left(2n+1,n^2,2\right)-3\left(cp_n^4+2cp_n^{22}\right)-t_{32,n}\equiv 1(\mathrm{mod}\,31)$
5) $q(2n-1,2,1)-p(2n-1,2n-1,1)+y(2n,2n-1,1)+2=t_{58,n}+t_{42,n}-36g_n$
6)

Pattern 2:

The assumption $u=X+(3k-1)T$ and $v=X-kT$ (7)

in (3) gives $X^2+k(3k-1)T^2=z^2$ (8)

Case 1:

$k(3k-1)\neq$ square (9)

Assume $z=a^2+k(3k-1)b^2$ (10)

From (8) and (10), we have, on factorization

$$X+i\sqrt{k(3k-1)}T=\left(a+i\sqrt{k(3k-1)}b\right)^2$$

$$X-i\sqrt{k(3k-1)}T=\left(a-i\sqrt{k(3k-1)}b\right)^2$$

On equating real and imaginary parts, in either of the above two equations we obtain

$$X=a^2-k(3k-1)b^2$$

$$T=2ab$$

On substituting X and T in (4) we get the values of u and v to be

$$u=a^2-3k^2b^2+kb^2+6abk-2ab$$

$$v=a^2-3k^2b^2+kb^2-2kab$$

Substituting u and v in (2) the values of x, y, p and q are given by

$$x=x(a,b,k)=2a^2-6k^2b^2+2kb^2+4abk-2ab$$

$$y=y(a,b,k)=8abk-2ab$$

$$p=p(a,b,k)=3a^2-9k^2b^2+3kb^2+10abk-4ab \qquad (11)$$

$$q=q(a,b,k)=a^2-3k^2b^2+kb^2+14abk-4ab$$

Thus (10) and (11) represent the non-zero distinct integer solutions of (1)

Properties:

1) $y(a,a,1)=2(t_{8,a}+g_a+1)$
2) $q(1,1,k)-p(1,1,k)-2g_k+1$
3) $\frac{x(1,b,1)}{2}=1-t_{6,b}$
4) $p(a,1,1)-q(a,1,1)+y(a,1,1)+4=t_{18,a}-t_{14,a}$
5) $y(n+1,n+2,2)+x(n+1,n+2,2)=ct_{28,n}+ct_{8,n}+31g_n+87$

Case 2:

Choosing $k(3k-1)=w^2$ in (5), it is satisfied by

$X=2w^2RS$, $T=w(R^2-S^2)$ and $z=w(R^2+S^2)$

Substituting X and T in (4) we get the values of u and v to be

$$u=2w^2RS+3kwR^2-3kwS^2-wR^2+wS^2$$
$$v=2w^2RS-kwR^2+kwS^2$$

Substituting u and v in (2) we get the values of x, y, p and q are given by

$$\left.\begin{aligned} x&=x(R,S,w,k)=4w^2RS+2kwR^2-2kwS^2-wR^2+ws^2\\ y&=y(R,S,w,k)=4kwR^2-4kwS^2+wR^2+wS^2\\ p&=p(R,S,w,k)=6w^2RS+5kwR^2-5kwS^2-2wR^2+2wS^2\\ q&=p(R,S,w,k)=2w^2RS+7kwR^2-6wS^2-2wR^2\end{aligned}\right\} \quad (12)$$

Thus (10) and (12) represent the non-zero distinct integer solutions of (1)

Properties:

1) $z(2^n,2^n,1,1)=2(mer_{2n}+1)$
2) $p(1,1,2^n,1)-x(1,1,2^n,1)=2(jal_{2n}-1)$
3) $q(2^n,1,1,1)=Tha_{2n}+ky_n+mer_{2n}-3$
4) $x(1,1,w,k)-y(1,1,w,k)+2q(1,1,w,k)+1=t_{10,w}-g_w+7wg_k$
5) $p(2,1,1,(n+1)^2)+q(2,1,1,2n)-2t_{17,n}\equiv 1(\mathrm{mod}\,9)$

Pattern 3:

Rewrite (5) as

$$z^2-X^2=k(3k-1)T^2 \quad (13)$$

$$(z+X)(z-X)=k(3k-1)T^2 \quad (14)$$

Case 1:

(14) can be written as the system of double equations as

$$z+X=(3k-1)T \quad (11)$$

$$z-X=kT \tag{12}$$

Which is satisfied by

$$z=(4k-1)t,\ X=(2k-1)t,\ T=2t$$

On substituting X and T in (7) we get the values of u and v which are given by

$$u=(8k-3)t,\ v=(2k+1)t$$

On substituting u and v in (2) we get the values of x, y, z, p and q. The non-zero distinct integrals values of x, y, z, p and q satisfying (1) are given by

$$x=x(k,t)=(10k-2)t$$
$$y=y(k,t)=(6k-4)t$$
$$z=z(k,t)=(4k-1)t$$
$$p=p(k,t)=(18k-5)t$$
$$q=q(k,t)=(14k-7)t$$

Properties:

1) $x(n^2,n)+y(n,n)+n=10cp_n^6+t_{14,n}$
2) $p(n,n^2)+q(n,n)+5=6cp_n^{18}+2ct_{11,n}-3g_n$
3) $x(n,1)y(n,1)-(2t_{62,n}+3g_n)=11$
4) $p(n,n^2)-2(n+1,n)+z(n^2,2)-6cp_n^{23}+2ct_{25,n}\equiv 1 \pmod{12}$
5) $q(n,2n^2+1)-3y(n+1,n)-6cp_n^{20}-3cp_n^{16}\equiv 7 \pmod{37}$

Case 2:

Also, (10) can be written as the system of double equations as

$$z+X=(3k^2-k)T \tag{13}$$

$$z-X=T \tag{14}$$

Applying the same procedure as in case 1 we get the non-zero distinct integrals values of x, y, z, p and q satisfying (1) are given by

$$x=x(k,t)=(6k^2-3)t$$
$$y=y(k,t)=(4k^2-1)t$$
$$z=z(k,t)=(3k^2-k+1)t$$
$$p=p(k,t)=(9k^2+2k-5)t$$
$$q=q(k,t)=(3k^2+6k-3)t$$

Properties:

1) $x(n,1)y(n,1)-\left(t_{50,n^2}+t_{12,n}\right)\equiv 3 \pmod 4$

2) $z(2^n,1)-Tha_{2n}+mer_n=1$

3) $p(n+1,2)-t_{38,n}\equiv 17(\bmod 23)$

4) $42\left(7p(k,t)-2q(k,t)+2z(k,t)+27\right)$ = Nasty Number

5) $2y\left(2^n,n\right)-z\left(2^n,n\right)-4\left(wo_{2n}+cul_{2n}\right)+Tha_{2n}-mer_n\equiv -1(\bmod 2)$

Case 3:

In addition to case 2, rewrite (14) as the pair of equations as

$$z+X=T^2 \quad (16)$$

$$z-X=3k^2-k \quad (17)$$

Repeating the procedure as in case 1, we get the non-zero distinct integrals values of x, y, z, p and q satisfying (1) are given by

$$x=x(k,t)=4t^2-12k^2+2k+8kt-2t$$

$$y=y(k,t)=16kt-2t$$

$$z=z(k,t)=2t^2+6k^2-k$$

$$p=p(k,t)=6t^2-18k^2+6k+20kt-4t$$

$$q=q(k,t)=-6k^2+2k-4t+4kt$$

Properties:

1) $x(k,1)-y(k,1)+ct_{24,k}-3g_k=2$

2) $q\left(2^n,1\right)+3carl_n+Tha_{2n}+8=0$

3) $p(1,t)-t_{14,t}\equiv -12(\bmod 21)$

4) $y(n+1,n+1)+2q(n,1)-ct_{6,n}-p_n\equiv 1(\bmod 8)$

5) $z(2n,1)\,y(n,1)-8q(n,1)=384cp_n^6+2(g_n-1)$

III. CONCULSION

It is worth to note that in (2), the transformations for z and w may be considered as $p=2uv+1$ and $q=2uv-1$. For this case, the values of x, y and z are the same as above where as the values of p and q changes for every pattern. To conclude one may consider biquadratic equation with multivariables(≥5)and search for their non-zero distinct integer solutions along with their corresponding properties.

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Blue Eyes Intelligence Engineering
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